Nonlinear Dyn (2017) 87:2563–2575
DOI 10.1007/s11071-016-3211-5

ORIGINAL PAPER

Dynamics of fault motion in a stochastic spring-slider model with varying neighboring interactions and time-delayed coupling

Srđan Kostić · Nebojša Vasović · Igor Franović · Kristina Todorović · Vladimir Klinshov · Vladimir Nekorkin

Received: 9 December 2015 / Accepted: 13 November 2016 / Published online: 26 November 2016

Abstract We examine dynamics of a fault motion by analyzing behavior of a spring-slider model composed of 100 blocks where each block is coupled to a varying number of $2K$ neighboring units ($1 \leq 2K \leq N$, $N = 100$). Dynamics of such model is studied under the effect of delayed interaction, variable coupling strength and random seismic noise. The qualitative analysis of stability and bifurcations is carried out by deriving an approximate deterministic mean-field model, which is demonstrated to accurately capture the dynamics of the original stochastic system. The primary effect concerns the direct supercritical Andronov–Hopf bifurcation, which underlies transition from equilibrium state to periodic oscillations under the variation of coupling delay. Nevertheless, the impact of delayed interactions is shown to depend on the coupling strength and the friction force. In particular, for loosely coupled blocks and low values of friction, observed system does not exhibit any bifurcation, regardless of the assumed noise amplitude in the expected range of values. It is also suggested that a group of blocks with the largest displacements, which exhibit nearly regular periodic oscillations analogous to coseismic motion for system parameters just above the bifurcation curve, can be treated as a representative of an earthquake hypocenter. In this case, the distribution of event magnitudes, defined as a natural logarithm of a sum of squared displacements, is found to correspond well to periodic (characteristic) earthquake model.

Keywords Mean-field approximation · Spring-block model · Seismic noise · Time delay · Coupling strength · Periodic earthquake model

S. Kostić (✉)
Department for Scientific Research and Informatics, Institute for Development of Water Resources "Jaroslav Černi", Jaroslava Černog 80, Belgrade 11226, Serbia
e-mail: srdjan.kostic@jcerni.co.rs

N. Vasović
Department of Applied Mathematics and Informatics, Faculty of Mining and Geology, University of Belgrade, PO Box 162, Belgrade, Serbia

I. Franović
Scientific Computing Lab, Institute of Physics Belgrade, University of Belgrade, Pregrevica 118, Belgrade 11080, Serbia

K. Todorović
Department of Mathematics and Physics, Faculty of Pharmacy, University of Belgrade, Vojvode Stepe 450, Belgrade, Serbia

V. Klinshov · V. Nekorkin
Institute of Applied Physics, Russian Academy of Sciences, 46 Ulyanov Street, Nizhny Novgorod, Russia

1 Introduction

From the purely mechanical viewpoint, the dynamics along an active fault could be described by a stick-slip motion of a spring-slider model complemented by an appropriate rate- and (or) state-dependent friction law [1–4]. For such a setup, a sudden drop of shear strength or an increase in block velocity is interpreted as an onset of seismic motion, while the magnitude of an event is defined as a natural logarithm of a superthreshold displacement of a block for a simple mono-block model, or a sum of superthreshold displacements of blocks involved in the event in case of a complex multi-block model [5]. The distribution of magnitudes should follow a power-law behavior, viz. the Gutenberg–Richter law, in order for a spring-block model to simulate the real observed fault dynamics. However, mono-block models [6,7] typically do not generate irregular time series of displacements unless some additional ingredients are introduced, including the friction lag [8], transient acceleration changes [9], the seismic noise or some particular forms of parameter perturbation [10]. Nevertheless, irregular distribution of main events is inherent for multi-block models, where a different number of moving blocks may participate in each event. For instance, De Sousa Vieira [6] considered the dynamics of two- and three-block models, having found that the relevant complex behavior is associated with the emergence of deterministic chaos obtained by fine tuning of control parameters. Similarly, Erickson et al. [11] analyzed the dynamics of a one-dimensional Burridge–Knopoff model, showing that it also exhibits both periodic and chaotic motion, whereby the transition to chaos is size dependent. In the present paper, we also examine the dynamics of a multi-block fault model, under the effect of two additional factors that impact the local block dynamics.

In particular, we assume that the main source of dynamical instability in a spring-block system, corresponding to the onset of seismic activity along a real fault, derives from the delayed interaction between the blocks. Introduction of such a delay has first been suggested by Burridge and Knopoff [1], who discussed this type of interaction in their original model comprised of 10 blocks. In their study, the idea has been that the stress transfer between the first and the last few blocks of the array is mediated by a group of central blocks, involving a time delay of the order of the adopted viscous time constant. In our study, this effect is included by explicitly assuming the delayed interaction between the coupled units.

Apart from the inclusion of time delay, we also investigate the effect of the range of interactions on the fault dynamics. In particular, we consider the cases from the nearest-neighbor interactions up to a globally connected assembly of blocks. The intention is to examine whether and how the range of interactions influences the scenario of transition to seismic motion. This effect is analyzed in the presence of a background seismic noise, which we assume to be the white noise, the point corroborated by the real observed data [12,13] and the previous studies of stick-slip motion [14].

The considered setup of a spring-block model of a fault motion, which involves the effects of coupling delay, the seismic noise as well as $2K$ nearest-neighbor interactions, is substantially distinct from the limit case of a single block motion discussed in our previous study [8], or the model involving only the nearest-neighbor interactions [15], used to describe the case of a thin isolated plate subjected to a friction force and driven by a shear force. Some previous studies have considered more complex spatiotemporal interactions of blocks [16–18], but without including the effects of delayed interaction and seismic noise. Our aim is to analyze the stability and bifurcations of the fault dynamics under the effect of the two latter factors. To this end, we shall derive a mean-field approximate model for the collective motion of blocks [19–21].

The paper is organized as follows. In Sect. 2, we describe the model of a fault motion comprised of delay-coupled blocks influenced by seismic noise. We also introduce the mean-field approximation of the system's collective dynamics. In Sect. 3 are presented the results of local bifurcation analysis carried out for the approximate system, together with the comparison with the behavior of the starting stochastic system. Section 4 concerns the analysis of magnitude distribution of main events for a group of blocks with the largest displacements. A summary of our main results and directions for further research are provided in Conclusions.

2 Model derivation

We consider a system comprised of N blocks, whereby each block i interacts with $2K$ nearest neighbors. Thus, the fault is represented as a one-dimensional array of blocks where a given block i is coupled to K of its neighbors on each side. Local dynamics is given by:

$$\begin{aligned}\dot{u}_i &= v_i \\ \dot{v}_i &= -u_i + \Phi\left(v_i + v_0\right) - \Phi(v_0) \\ &\quad + \frac{C}{N}\sum_{j\in J}\left(u_{i+j}\left(t-\tau\right) - u_i\right) + \sqrt{2D}\xi_i\left(t\right),\end{aligned} \quad (1)$$

where J denotes the set of indices $J = \{-K, \ldots, K\}\backslash\{0\}$. Interaction is characterized by the coupling strength

C and the delay τ, which is assumed to be uniform. Parameter v_0 denotes the pulling velocity of the upper moving plate. $\xi_i(t)$ are independent Gaussian white noise terms, such that $\langle \xi_i \rangle = 0$, $\langle \xi_i \xi_j \rangle = \delta_{ij}$ holds. In general, seismic noise may derive from various sources, including small-scale faulting, as well as different irregularities and inhomogeneities, but there may also be random perturbations of undefined origin [22,23]. Note that coherent noise is hardly expected to occur at seismogenic depth, since it typically arises from the reflection of seismic waves, ground rolls or traffic noise. A detailed background regarding model (1) is provided in “Appendix”.

It has to be emphasized that model (1) represents a one-dimensional array of blocks, where blocks at the end of the array are connected to neighboring blocks only on one side.

One should specify the friction force Φ, which is in (1) for simplicity given in general form. We assume that Φ conforms to a rate-dependent friction law:

$$\Phi(v) = \mu_0 - a \ln(v), \tag{2}$$

where μ_0 is a steady-state friction (whose values needs to be adequately chosen so as to secure the proper action of friction force) and a represents a material property which depends on the temperature and pressure conditions. Such dependence of friction on slip rate is qualitatively supported by the recent laboratory findings [24,25].

In the absence of coupling delay and seismic noise ($\tau = 0, D = 0$), model (1) has a unique stable equilibrium (for the assumed distance between neighboring blocks in initial position of the order of magnitude 10^{-5}) given by $(u_i, v_i) = (0, 0)$.

In principle, the collective dynamics of the multiblock model can be described by introducing the macroscopic variables $u = \frac{1}{N}\sum_{i=1}^{N} u_i$, $v = \frac{1}{N}\sum_{i=1}^{N} v_i$. In order to be able to analyze dynamics of the starting stochastic model, we need to derive its deterministic approximation, which will qualitatively describe its dynamics, and which will enable conduction of local bifurcation analysis. The method we apply consists in deriving a deterministic mean-field model [26] for the macroscopic behavior of system (1), whereby the noise intensity features as an additional bifurcation parameter. Within such a framework, the collective dynamics is described in terms of the means, viz. the assembly-averaged displacement m_u and the average velocity m_v, as well as the associated variances and the covariance. The ensuing mean-field model is amenable to local bifurcation analysis and as such may serve to qualitatively describe the stability and bifurcations of the starting stochastic system.

The detailed derivation of the mean-field model is presented in “Appendix”. As discussed there, the mean-field model may be presented as a system of two delay-differential (deterministic) equations describing the evolution of the means:

$$\begin{aligned} \dot{m}_u(t) &= m_v(t) \\ \dot{m}_v(t) &= -m_u(t) + a\ln(v_0) - a\ln(m_v + v_0) \\ &\quad + \frac{1}{2}\frac{D}{(m_v + v_0)} + \frac{2KC}{N}\left(m_u(t-\tau) - m_u(t)\right) \end{aligned} \tag{3}$$

In the next section are provided the results of bifurcation analysis for the model (1), together with a brief discussion on the effects of each of the system parameters. The numerical simulations of the starting stochastic system (1) are carried out for the array of $N = 100$ blocks using the Runge–Kutta fourth-order algorithm with a time step $\Delta t = 0.001$.

3 Local bifurcation analysis of the mean-field model

Mean-field model (3) has a unique stable stationary solution $(m_u, m_v) = (\frac{D}{2v_0}, 0)$. We have performed the stability and bifurcation analysis of model (3) analytically. In particular, system (3) is linearized around the fixed point by assuming that the deviations are of the form $\delta m_u(t) = Ae^{\lambda t}$, $\delta m_v(t) = Be^{\lambda t}$, $\delta m_u(t-\tau) = Ae^{\lambda(t-\tau)}$. This yields a set of algebraic equations for the coefficients A and B, whereby the condition for the existence of a nontrivial solution is given by the characteristic equation:

$$\lambda^2 + \frac{\lambda}{v_0}\left(a + \frac{D}{2v_0}\right) + 1 + \frac{2KC}{N}(1 - e^{-\lambda\tau}) = 0. \tag{4}$$

Bifurcations of the stationary state take place for the parameter values where the roots of characteristic equation cross the imaginary axis. Given that $\lambda = 0$ is not the solution of (4), we look for the pure imaginary roots of the form $\lambda = i\omega$, adopting that ω is real and positive. Having substituted for λ in (4), one separates the

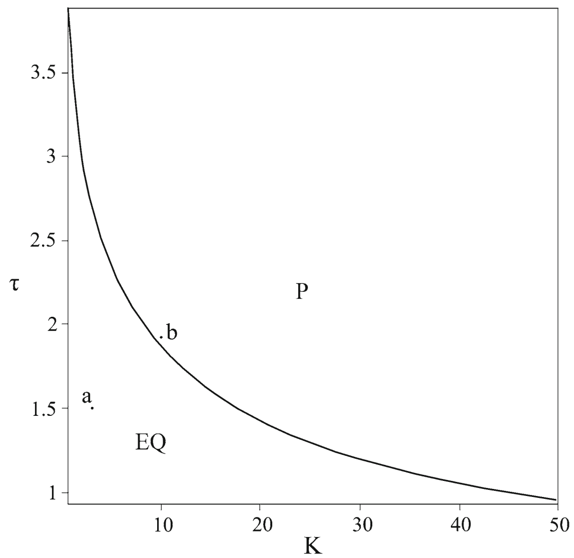

Fig. 1 $\tau(K)$ bifurcation curve describing the destabilization of equilibrium in the mean-field model (3) via Andronov–Hopf bifurcation. The remaining system parameters are fixed at $C = 5$, $D = 0.0001$, $a = 0.1$, $v_0 = 1.2$. Qualitatively similar diagrams are obtained for other values of C and D. *EQ* and *P* denote the equilibrium state and periodic oscillations, respectively. Corresponding time series for mean displacement of systems (1) and (3) at points a and b are provided in Fig. 2

real and the imaginary part and equates them both with zero. After some algebra, we arrive at the expression for the critical coupling delay:

$$\tau = \frac{1}{\omega} arctg \frac{\frac{\omega}{v_0}\left(a + \frac{D}{2v_0}\right)}{\omega^2 - \left(1 + \frac{2KC}{N}\right)} \tag{5}$$

Further analysis shows that the mean-field model (3) undergoes Andronov–Hopf bifurcation. Bear in mind that (5) actually defines multiple branches of Hopf bifurcation curves given by $\tau + j\pi/\omega$, where $j = 0, 1, 2, \ldots$. In the present study, we focus only on the first bifurcation curve, since the starting stochastic system (1) above the bifurcation curve exhibits behavior characteristic for strongest earthquakes, as explained further below.

In order to facilitate an easier comparison between the dynamics of the approximate model and the starting stochastic system, which is simulated for $N = 100$, we have also fixed $N = 100$ in (5) and plotted the corresponding $\tau(K)$ bifurcation curve, see Fig. 1. Naturally, the relevant range of K values is then $K \in [1, 50]$. Also note that the analysis here is confined to values of coupling delay which are of the order of the oscillation period for the mean-field model (3) just above the bifurcation curve ($T \approx 3.5$).

Qualitative validity of the above results is verified numerically by demonstrating that the starting stochastic system (1) and the mean-field model (3) exhibit qualitatively analogous dynamics, see Fig. 2. From Fig. 2, one may infer that the mean-field model in qualitative sense accurately reproduces the collective behavior of the original model (1). An important point is that we also find a quantitative agreement between the dynamics of the two systems in a sense that the respective oscillation frequencies of the starting stochastic and the approximate system above the bifurcation curve are quite closely matched. The difference in respective amplitudes of periodic motion is negligible, considering the fact that matching of the amplitudes could be found only in cases where the behavior of the mean-field model exhibits similar displacements as the original model comprised of a large number of individual blocks. It could be easily shown that such an outcome can be expected for the multi-block model made up of blocks coupled in the all-to-all fashion.

As shown in Fig. 1, our analysis indicates the occurrence of a supercritical Andronov–Hopf bifurcation from equilibrium state to periodic motion, which is induced by increasing the time delay in interaction among the coupled blocks. For relatively high values of τ, bifurcation can arise even in case of very small range of interactions ($K = 3$). Nonetheless, the results obtained also imply that the bifurcation may occur by increasing the number of interacting blocks for a constant value of coupling delay. From the seismological viewpoint, this means that onset of seismic motion could be induced solely by enlarging the active length of a seismogenic fault, provided there is a delayed interaction among different fault segments.

Results shown in Fig. 1 also imply that there is no bifurcation for time delay $\tau < 0.9$. In this case, system (1) remains in equilibrium state under increase in K, viz. coupling more blocks together does not lead to the emergence of the oscillations. From the seismological viewpoint, this means that effect of delayed interaction is essential for occurrence of periodic oscillations, i.e., seismic regime, which will be verified in the subsequent section.

Another interesting point is that for the case when block is coupled only to its nearest neighbors ($K = 1$), there is no bifurcation with the increase in time delay.

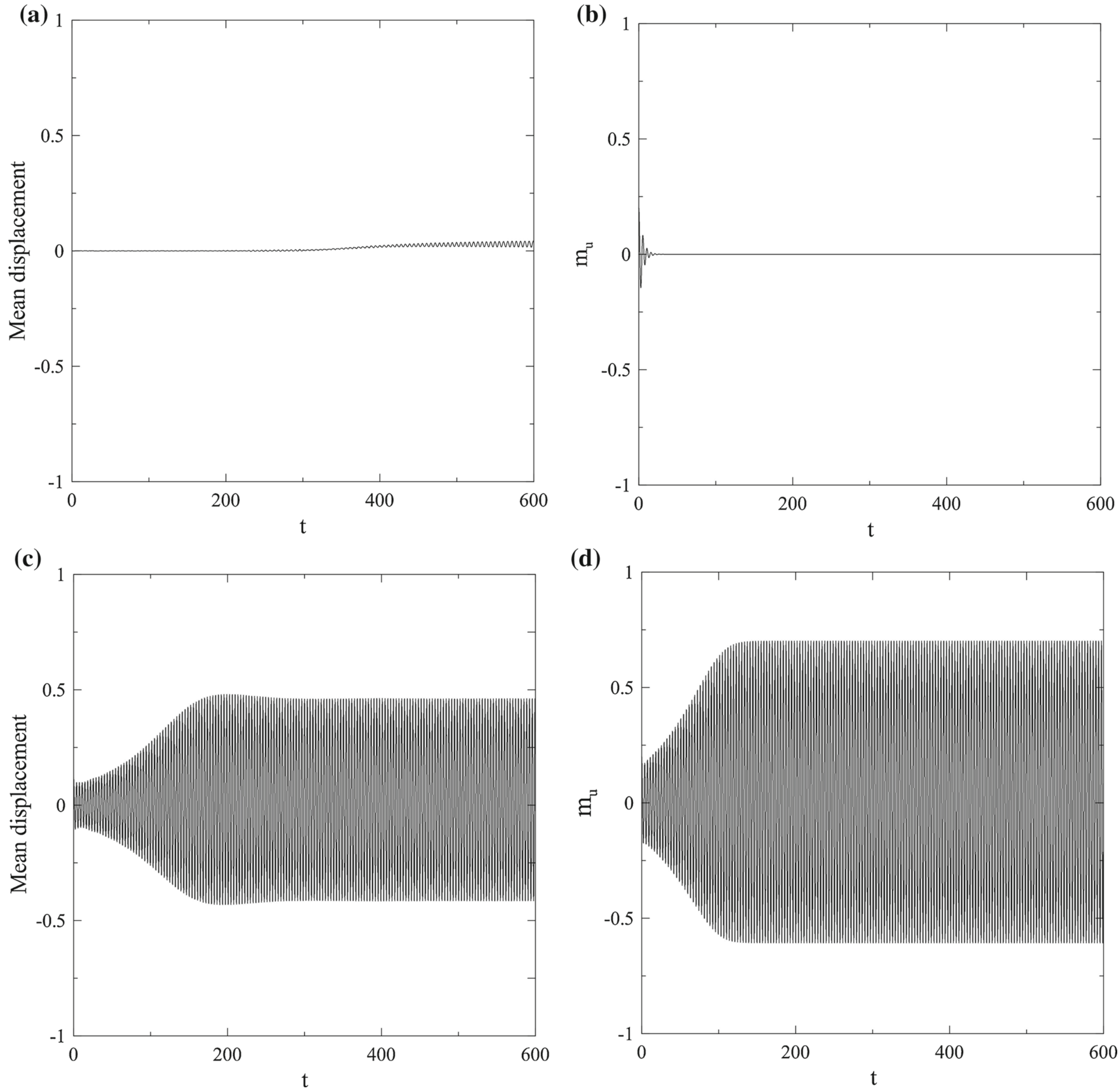

Fig. 2 Qualitative analogy between the dynamics of the starting stochastic system and the mean-field approximated model. The *left column* shows the time series of mean displacements of blocks for the starting stochastic system, whereas the *right column* refers to the approximated system. The top (*bottom*) row is obtained for parameter values corresponding to point a (point b) in Fig. 1. Initial conditions are set in vicinity of the equilibrium point. Equilibrium state is illustrated for $\tau = 1.5$ and $K = 3$, while periodic oscillations are illustrated for $\tau = 1.92$ and $K = 10$. Qualitatively similar diagrams of periodic oscillations are obtained for other values of K and τ just above the bifurcation curve. The remaining parameters are fixed at $C = 5$, $D = 0.0001$, $a = 0.1$, $\upsilon_0 = 1.2$

Hence, it turns out that for such simple models, the effect of delayed interaction is negligible.

One should note that previous analysis is conducted for the constant values of parameters C, D and a. Given the fact that these parameters could significantly affect the dynamics of fault motion, it is of special interest to analyze how the dynamics of the original system changes under their variation.

3.1 Effect of seismic noise

Effect of seismic noise on dynamics of system (1) is negligible for the range of values relevant from the seismological viewpoint, i.e, the common ratio of coseismic slip rate versus seismic noise is in the range 10^{-2}–10^{-7}. Nonetheless, in the context of investigating the noise effects, we have further examined the possibility for the occurrence of stochastic bifurcation in system (1), which conforms to the scenario where the transition to oscillatory motion is induced by increasing the noise intensity.

However, having examined the average amplitudes of a sufficient number of realizations of system (1) below and above the bifurcation curve, we have observed no significant change for values of seismic noise up to 0.1. Thus, the qualitative impact of noise on system dynamics can be considered secondary to that of coupling delay. Also, for relatively high values of seismic noise and for small K, introduction and increase in time delay does not give rise to bifurcation. In other words, noise may then suppress the onset of the periodic oscillations. For instance, if the level of seismic noise is $D = 1$, while the other parameters are fixed as shown in Fig. 1, at least $K = 6$ is required for the Andronov–Hopf bifurcation to emerge by increasing the time delay. However, from a seismological viewpoint, such a case cannot be expected in real conditions in Earth's crust, so these phenomena could be interesting solely from the theoretical viewpoint.

3.2 Effect of coupling strength

In qualitative terms, the impact of coupling strength C on dynamics of a spring-block model with time delay is similar to the effect of K. In particular, bifurcation in the starting stochastic system (1) occurs solely by increasing the coupling strength provided there is interaction delay between the units, see Fig. 3. Also, in analogy with the effect of K, one finds a delay threshold below which the increase in coupling strength does not induce any dynamical change. Nevertheless, for smaller values of C, increase of time delay may or may not induce bifurcation, depending on the particular K value. In other words, for small C, there exists a threshold K value below which increase in τ cannot give rise to Andronov–Hopf bifurcation. For instance, when $C = 0.5$, each block has to be coupled with at least 15 blocks on each side in order for the system to undergo the Andronov–Hopf bifurcation with the increase in τ.

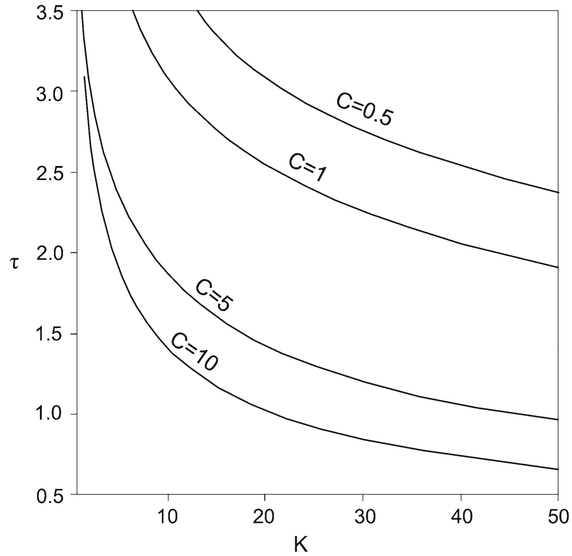

Fig. 3 Family of τ(K) Andronov–Hopf bifurcation curves of the mean-field model (3) for different values of coupling strength C. The remaining parameters are held constant at $D = 0.0001$, $a = 0.1$, $v_0 = 1.2$. Qualitatively similar diagrams are obtained for other values of friction parameter a

3.3 Effect of friction

From the seismological viewpoint, it is interesting to analyze the effect of friction parameter a on the dynamics of system (1). The results obtained indicate the occurrence of an inverse supercritical Andronov–Hopf bifurcation, induced solely by decreasing the friction (Fig. 4). Such a change of friction is expected in real conditions, because friction usually decreases with the increase in fault motion velocity. Moreover, this suggests that for the fault in equilibrium state, a reduction of friction along the fault zone due to effect of pore water or similar may give rise to bifurcation, provided there are delayed interactions among different parts of a fault, i.e., different fault segments.

4 Distribution of main events

The results obtained indicate a transition from equilibrium state to periodic motion may arise by increasing

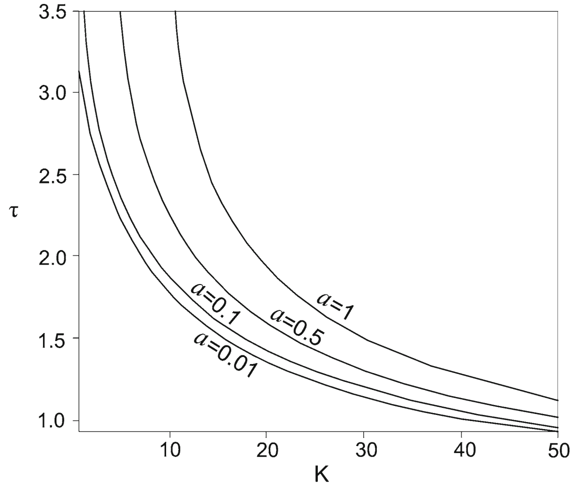

Fig. 4 Family of τ(K) Andronov–Hopf bifurcation curves describing the destabilization of equilibrium of the mean-field model for different values of friction parameter a. The remaining parameters are fixed at $D = 0.0001$, $C = 5$, $v_0 = 1.2$. Qualitatively similar diagrams are obtained for other values of C

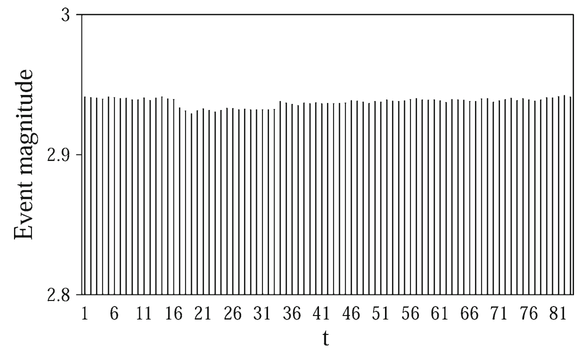

Fig. 5 Time series of displacement sums for a group of blocks with the largest displacements in the starting stochastic system (1). Initial conditions are set near the equilibrium point. The parameter values are fixed at $\tau = 1.92$, $K = 10$, $C = 5$, $D = 0.0001$, $a = 0.1$, $v_0 = 1.2$

the time delay, or due to increase in coupling strength and/or decrease in friction force under the condition that there exists a coupling delay. From the purely seismological viewpoint, if we relate magnitude of a seismic event to displacement of system (1), it appears that regular periodic motion could not be considered as an example of a realistic scenario given that the sequence of earthquakes should obey power-law behavior. In particular, a necessary condition for the Gutenberg–Richter scaling law to be satisfied is that the sequence of seismic events is irregular. Such a dynamical regime could emerge due to effect of seismic noise or the co-effect of seismic noise and some global attractor when the system is near the bifurcation curve, but still in the equilibrium state [27]. However, the results of our analysis here indicate that the sole effect of seismic noise is insufficient to generate high-amplitude irregular oscillations (higher than the noise level, at least). Nonetheless, we have also found no global attractor for different values of initial conditions away from the equilibrium point. In light of these arguments, the interpretation of the results so far requires additional attention. In particular, a better understanding of the relation to actually observed behavior of the real faults may be gained by looking into the dynamics of a group of coupled blocks with the largest displacements in the considered array (1). This group of blocks we take to correspond to a hypocenter of a seismic event. In the present case, magnitude of such an event M is defined as a natural logarithm of a sum of squares of displacements u_i:

$$M = \ln\left(\sum_{i=1}^{N}(u_i)^2\right) \tag{6}$$

where N is the number of coupled blocks with the largest displacements in system (1). This definition is similar to the proposal of Kawamura et al. [5], but with a small change regarding the fashion in which the displacement sum is calculated. In particular, we take the sum of squared displacement for each block as a measure of accumulated potential energy released during a single event. This way of calculating the stored potential energy corresponds to our initial assumption on elastic springs connecting the blocks in system (1).

In Fig. 5 is shown the regular sequence of displacement sums for a group of blocks with the largest displacements. Analysis is conducted for the state of system (1) just above the bifurcation curve, as shown in Fig. 2c. These events are regularly spaced, with approximately the same magnitude. Therefore, the proposed model (1) can be regarded as a periodic or characteristic earthquake model [28,29], which has already been observed in the real conditions in the Earth's crust. Indeed, the occurrence of earthquakes with the largest magnitudes along Nankai megathrust or Parkfield section of the San Andreas fault is considered to be nearly periodic [30].

5 Conclusions

The main goal of the present study has been to examine the possibility of inducing seismic motion along the real fault in case where fault segments of different sizes are active, under the assumption of a delayed interaction between the fault constituents, and including the impact of background seismic noise. The stability analysis, as well as the qualitative analysis of bifurcations exhibited by the given fault model, is carried out by considering the appropriate mean-field model. In particular, beginning from the original system of *2N* coupled stochastic delay-differential equations, we have derived a system comprised of only of two deterministic delay-differential equations for the approximate system.

The dynamics of the starting stochastic and the approximate model are demonstrated to be qualitatively similar. Another important point concerns the quantitative agreement between the two systems, in a sense that the oscillation frequency of the mean-field model matches quite closely the one observed for the stochastic system. Nevertheless, the oscillation amplitudes are the same only in case of globally coupled blocks in the starting stochastic system (1). In other instances, corresponding to smaller K, the amplitudes of the mean-field model may be up to 1.5 times higher than the amplitudes of periodic oscillations in the system (1), which does not have significant impact on our results from the viewpoint of seismology. In particular, we are only interested in type of oscillations (regular or irregular), while their amplitude is irrelevant, since the models is dimensionless; hence, there is no direct analogy with the real observed earthquake data.

Results of our research indicate that the onset of the oscillations in the starting stochastic system (1), which is shown to qualitatively reflect the seismic motion, can be characterized by the direct supercritical Andronov–Hopf bifurcation of the mean-field model. In the present study, we have focused only on the first bifurcation curve, describing the destabilization of equilibrium, whereas the relevant values of coupling delay have been determined according to the oscillation period of system (1) just above the bifurcation curve.

Regarding the individual effect of the introduced parameters, the main finding is that the transition in system (1) from equilibrium state to periodic oscillations is primarily affected by the coupling delay. However, the impact of the delayed interactions is found to strongly depend on the coupling strength. In particular, in case of ”weak” coupling, the increase in interaction delay cannot generate the transition to limit cycle. In other words, the model comprised of loosely coupled blocks cannot exhibit complex dynamics. A similar effect, though less expressed, is observed for the impact of friction. In particular, for lower values of friction, system (1) can exhibit the transition from equilibrium state to periodic motion for smaller K. As for the effect of seismic noise, the results obtained imply a less significant role in the dynamics of system (1) for the values consistent with the real observed data (ratio of seismic noise to velocity should be smaller than 10^{-2}). The impact of strong seismic noise (e.g., $D \approx 1.0$) is similar to the effect of high values of coupling strength and friction. Apparently, for such a large seismic noise, there is a certain K threshold above which the system can exhibit Andronov–Hopf bifurcation by increasing the coupling delay.

From the seismological perspective, an important point is that we propose a new definition of an earthquake magnitude for the considered class of spring-block models. In particular, we define magnitude of an event as a natural logarithm of sum of squared displacements for a group of blocks with the largest displacements. Our results indicate that these sequences of events are regular, whereby the distribution of the number of events in dependence of their magnitude corresponds to the occurrence of the strongest (characteristic) earthquakes, as the ones recorded along the Nankai megathrust or Parkfield section of San Andreas fault.

By analyzing dynamics of the group of blocks with largest displacements, we partially captured local dynamics of the real model, which also turned out to be regular, similarly to the mean displacement of the whole array of blocks. Significant deviation from the regular periodic oscillations could be expected for the blocks at the end of the array, which are only coupled to blocks at one side of the ”chain.” However, dynamics of these blocks is irrelevant from the viewpoint of seismology, since their displacements are quite low and are not connected directly to the earthquake ”hypocenter” (i.e., to the group of blocks with the largest displacements).

Though the present analysis provides a clear insight on the role of delayed interaction, seismic noise and coupling strength on the collective dynamics of the multi-block fault model, one should still note that the friction law (2) is assumed in a simplified way, neglecting the potential impact of the state of the contact surface. Moreover, we have analyzed only the case of a one-dimensional spring-slider model, while in real conditions the fault motion represents a spatiotemporal problem, the point which certainly has to be addressed in future research.

Acknowledgements This research has been supported by the Ministry of Education, Science and Technological development of the Republic of Serbia, Contracts No. 176016 and 171017, by the Russian Foundation for Basic research under grants No. 14-02-00042, 15-02-04245 and 15-32-50402, and by the Ministry of Education and Science of the Russian Federation, Agreement No. MK-8460.2016.2.

Appendix: Model of local block dynamics

Present research on earthquake fault motion is based on the analysis of a non-dimensional mono-block model, introduced in [6]:

$$\ddot{U}(t) = -U(t) + \Phi\left(\dot{U}(t)\right) + \nu_0 t, \tag{7}$$

where variable U represents the block displacement, $\dot{U}$ is the velocity of the block (defined in the standing reference frame), $\ddot{U}$ is the block acceleration, ν_0 is the dimensionless pulling speed and t is time variable. Friction force Φ is assumed to be rate dependent. The unstable equilibrium around which the orbit of a block moves in phase space is given by:

$$U_e(t) = \nu_0 t + \Phi(\nu), \tag{8}$$

which is determined by setting $\ddot{U} = 0$ and $\dot{U} = \nu_0$ in (7).

By introducing U_1 and U_2 to denote the displacement and the velocity of a single block, (7) may be rewritten as

$$\begin{aligned}\dot{U}_1(t) &= U_2(t)\\ \dot{U}_2(t) &= -U_1(t) + \Phi(U_2(t)) + \nu_0 t.\end{aligned} \tag{9}$$

Having applied the coordinate transformation:

$$\begin{aligned}U_{1new}(t) &= U_1 - U_e(t) = U_1(t) - (\nu_0 t + \Phi(\nu)),\\ U_{2\text{new}}(t) &= U_2(t) - \nu_0,\end{aligned} \tag{10}$$

and switching back to old notation, one arrives at the following system of equations for the dynamics of a single block:

$$\begin{aligned}\dot{U}_1(t) &= U_2(t)\\ \dot{U}_2(t) &= -U_1(t) + \Phi(U_2(t) + \nu_0) - \Phi(\nu_0).\end{aligned} \tag{11}$$

Beginning from (11), one can derive the spring-slider model for N interconnected blocks given by the system (1) in the main text, cf. Sect. Model derivation.

Derivation of the mean-field approximated model

The mean-field model characterizes the system's behavior in terms of the means:

$$m_u = \langle u_i \rangle = \frac{1}{N}\sum_i u_i, \quad m_v = \langle v_i \rangle = \sum_i v_i, \tag{12}$$

the associated variances:

$$\begin{aligned}s_u &= \left\langle \delta u_i^2 \right\rangle = \left\langle (u_i - m_u)^2 \right\rangle = \left\langle u_i^2 \right\rangle - m_u^2\\ s_v &= \left\langle \delta v_i^2 \right\rangle = \left\langle (v_i - m_v)^2 \right\rangle = \left\langle v_i^2 \right\rangle - m_v^2\end{aligned} \tag{13}$$

and the covariance

$$\begin{aligned}U &= \langle \delta u_i \delta v_i \rangle = \langle (u_i - m_u)(v_i - m_v) \rangle\\ &= \langle u_i v_i \rangle - m_u m_v.\end{aligned} \tag{14}$$

Equations for the evolution of the means may be derived as follows:

$$\begin{aligned}\dot{m_u} &= \frac{1}{N}\sum_i \dot{u_i} = \frac{1}{N}\sum_i v_i = m_v\\ \dot{m_v} &= \frac{1}{N}\sum_i \dot{v_i}\\ &= \frac{1}{N}\sum_i \Bigg\{ -u_i - \Phi(\nu_0) + \Phi(v_i + \nu_0)\\ &\quad + \frac{C}{N}\sum_{j\in J}\left(u_{i+j}(t-\tau) - u_i\right) + \sqrt{2D}\xi_i(t) \Bigg\}\\ &= -m_u - \Phi(\nu_0) + \frac{1}{N}\sum_i \Phi(v_i + \nu_0)\\ &\quad + \frac{1}{N}\sum_i \frac{C}{N}\sum_{j\in J}\left(u_{i+j}(t-\tau) - u_i\right).\end{aligned} \tag{15}$$

The third term in equation (15) may be evaluated as follows. In the limit of large N, one may replace the summation by an appropriate integral $\frac{1}{N}\sum_i \Phi(v_i + \nu_0) \to \int P(v)\Phi(v + \nu_0)\,\mathrm{d}v$, where $P(v)$ is the probability density function. Now if all v variables are appropriately Gaussian distributed around $m_v(t)$, then $\Phi(v+v_0)$ may be written as $\Phi(v+v_0) \approx \Phi(m_v+v_0) + \Phi'(m_v+v_0)\,\delta v + \frac{1}{2}\Phi''(m_v+v_0)(\delta v)^2$. In the latter expression, δv denotes the deviation $\delta v = v - m_v$. The integral may then be estimated as:

$$\begin{aligned}
&\int P(v)\,\Phi(v+v_0)\mathrm{d}v \\
&= \Phi(m_v+v_0)\int P(v)\,\mathrm{d}v + \Phi'(m_v+v_0) \\
&\quad \int P(v)\,\delta v\mathrm{d}v + \frac{1}{2}\Phi''(m_v+v_0)\int P(v)\delta v^2\mathrm{d}v \\
&= \Phi(m_v+v_0) + \frac{1}{2}\Phi''(m_v+v_0)\,S_v
\end{aligned} \tag{16}$$

since, by definition $\int P(v)\,\mathrm{d}v \equiv 1$, $\int P(v)(v-\langle v\rangle)\,\mathrm{d}v = \langle v\rangle - \langle v\rangle \int P(v)\,\mathrm{d}v = 0$ and $\int P(v)(v-\langle v\rangle)^2\,\mathrm{d}v \equiv S_v$.

The fourth term in (15) can be handled as follows:

$$\begin{aligned}
&\frac{1}{N}\sum_i \frac{C}{N}\sum_{j\in J}\left(u_{i+j}(t-\tau) - u_i(t)\right) \\
&= \left\langle \frac{C}{N}\sum_{j\in J}\left(u_{i+j}(t-\tau) - u_i(t)\right)\right\rangle \\
&= \frac{2KC}{N}\left(\langle u_i(t-\tau)\rangle - \langle u_i(t)\rangle\right) = \\
&= \frac{2KC}{N}\left(m_u(t-\tau) - m_u(t)\right).
\end{aligned} \tag{17}$$

Proceeding from (15), one then obtains:

$$\begin{aligned}
\dot{m_v} &= -m_u - \Phi(v_0) + \Phi(m_v+v_0) \\
&\quad + \frac{1}{2}\Phi''(m_v+v_0)\,S_v \\
&\quad + \frac{2KC}{N}\left(m_u(t-\tau) - m_u(t)\right).
\end{aligned} \tag{18}$$

Let us now determine the equations for the dynamics of the variances. From the definition of S_u, it follows that

$$\begin{aligned}
\dot{S_u} &= \langle 2u_i\dot{u_i}\rangle - 2m_u\dot{m_u} = \langle 2u_iv_i\rangle - 2m_um_v \\
&= 2(U + m_um_v) - 2m_um_v = 2U
\end{aligned} \tag{19}$$

For S_v, one finds:

$$\begin{aligned}
\dot{S_v} &= \langle 2v_i\dot{v_i} + 2D\rangle - 2m_v\dot{m_v} = \Bigg\langle 2v_i\Bigg(-u_i \\
&\quad - \Phi(v_0) + \Phi(v_i+v_0) \\
&\quad + \frac{C}{N}\sum_{j\in J}\left(u_{i+j}(t-\tau) - u_i\right)\Bigg)\Bigg\rangle \\
&\quad - 2m_v\left(-m_u - \Phi(v_0) + \Phi(m_v+v_0)\right. \\
&\quad + \frac{1}{2}\Phi''(m_v+v_0)\,S_v + \frac{2KC}{N}\left(m_u(t-\tau)\right. \\
&\quad \left.\left. - m_u(t)\right)\right) + 2D = -2(U + m_um_v) \\
&\quad - 2m_v\Phi(v_0) + 2\langle v_i\Phi(v_i+v_0)\rangle \\
&\quad + \frac{2C}{N}\left\langle v_i\sum_{j\in J}\left(u_{i+j}(t-\tau) - u_i\right)\right\rangle \\
&\quad + 2m_um_v + 2m_v\Phi(v_0) - 2m_v\Phi(m_v+v_0) \\
&\quad - m_v\Phi''(m_v+v_0)\,S_v - 2m_v\frac{2KC}{N}\left(m_u(t-\tau)\right. \\
&\quad \left. - m_u(t)\right) + 2D,
\end{aligned} \tag{20}$$

where the term $\langle 2v_i\dot{v_i} + 2D\rangle$ presents the Itō derivative for the complex function v_i^2 of the stochastic variable v_i.

The third term in (20) can be estimated in a fashion similar to the second term in (15). In particular, one may write:

$$\begin{aligned}
&2\langle v_i\Phi(v_i+v_0)\rangle = 2\left\langle (m_v+\delta v_i)\left(\Phi(m_v+v_0)\right.\right. \\
&\quad \left.\left. + \Phi'(m_v+v_0)\,\delta v_i + \frac{1}{2}\Phi''(m_v+v_0)\,\delta v_i^2\right)\right\rangle \\
&= 2\left\langle m_v\Phi(m_v+v_0) + m_v\Phi'(m_v+v_0)\,\delta v_i\right. \\
&\quad + \frac{1}{2}m_v\Phi''(m_v+v_0)\,\delta v_i^2 \\
&\quad \left. + \Phi(m_v+v_0)\,\delta v_i + \Phi'(m_v+v_0)\,\delta v_i^2\right\rangle \\
&= 2\left(m_v\Phi(m_v+v_0) + \frac{1}{2}m_v\Phi''(m_v+v_0)\,S_v\right. \\
&\quad \left. + \Phi'(m_v+v_0)\,S_v\right).
\end{aligned} \tag{21}$$

Given that the deviations from the mean are expected to be small, we keep only the terms up to second order in deviations. Also, $\int m_v\Phi'(m_v)(v-\langle v\rangle)P(v)\,\mathrm{d}v = m_v\Phi'(m_v)(\langle v\rangle - \langle v\rangle) = 0$.

The fourth term in (20) can be calculated by implementing the quasi-independence approximation:

$$\begin{aligned}
&\frac{2C}{N}\left\langle v_i\sum_{j\in J}\left(u_{i+j}(t-\tau) - u_i\right)\right\rangle \\
&\quad = \frac{2C}{N}\left\langle \sum_{j\in J}\left(v_iu_{i+j}(t-\tau) - v_iu_i\right)\right\rangle
\end{aligned}$$

$$= \frac{2C}{N}\sum_{j\in J}\left\langle v_i u_{i+j}(t-\tau)\right\rangle - \frac{4KC}{N}\left\langle u_i v_i\right\rangle$$
$$= \frac{2C}{N}\sum_{j\in J}\left\langle v_i\right\rangle\left\langle u_{i+j}(t-\tau)\right\rangle$$
$$-\frac{4KC}{N}\left(U+m_u m_v\right)$$
$$= \frac{4KC}{N} m_v m_u(t-\tau) - \frac{4KC}{N}\left(U+m_u m_v\right). \quad (22)$$

From the strict mathematical viewpoint, one should note that ⟨vi(t) ui+j (t − τ)⟩ = ⟨vi(t)⟩ ⟨ui + j (t − τ)⟩ holds only for very large τ. This assumption is justified in the present case, since, according to Burridge and Knopoff [1], time delay between the neighboring group of seismically active elements (blocks) should be two orders of magnitude smaller than corresponding time constant for the moving blocks. On the contrary, in the present study, we examine the effect of time delay up to value of $\tau = 3.5$, i.e., of the order of the oscillation period for the mean-field model (3) just above the bifurcation curve ($T \approx 3.5$), which is considered to correspond to seismic regime of large events, as previously discussed in Sect. 4.

Inserting (21) and (22) into (20), one arrives at:

$$\dot{S_v} = -2U\left(1+\frac{2KC}{N}\right) + 2\Phi'\left(m_v+v_0\right)S_v + 2D \quad (23)$$

The equation for the dynamics of the covariance can be obtained as follows:

$$\dot{U} = \langle \dot{u_i} v_i\rangle + \langle u_i \dot{v_i}\rangle - \dot{m_u} m_v - m_u \dot{m_v}$$
$$= \left\langle v_i^2\right\rangle + \left\langle u_i\left(-u_i - \Phi(v_0) + \Phi(v_i+v_0)\right.\right.$$
$$\left.\left. + \frac{C}{N}\sum_{j\in J}\left(u_{i+j}(t-\tau)-u_i\right) + \sqrt{2D}\xi_i(t)\right)\right\rangle$$
$$- m_v^2 - m_u\left(-m_u - \Phi(v_0) + \Phi(m_v+v_0)\right.$$
$$+ \frac{1}{2}\Phi''(m_v+v_0)S_v + \frac{2KC}{N}\left(m_u(t-\tau)\right.$$
$$\left.\left. - m_u(t)\right)\right) \quad (24)$$

The two terms that have to be considered more carefully are $\langle u_i\Phi(v_i+v_0)\rangle$ and $\frac{C}{N}\left\langle u_i\sum_{j\in J}\left(u_{i+j}(t-\tau)-u_i\right)\right\rangle$. As for the former, one may write:

$$\langle u_i\Phi(v_i+v_0)\rangle = \left\langle (m_u+\delta u_i)\left(\Phi(m_v+v_0)\right.\right.$$
$$\left.\left. + \Phi'(m_v+v_0)\delta v_i + \frac{1}{2}\Phi''(m_v+v_0)\delta v_i^2\right)\right\rangle$$
$$= \left\langle m_u\Phi(m_v+v_0) + m_u\Phi'(m_v+v_0)\delta v_i\right.$$
$$\left. + \frac{1}{2}m_u\Phi''(m_v+v_0)\delta v_i^2\right\rangle +$$
$$+ \left\langle \Phi(m_v+v_0)\delta u_i + \Phi'(m_v+v_0)\delta u_i\delta v_i\right.$$
$$\left. + \frac{1}{2}\Phi''(m_v)\delta u_i\delta v_i^2\right\rangle. \quad (25)$$

In analogy to (21), one has $\left\langle m_u\Phi'(m_v+v_0)\delta v_i\right\rangle = 0$ and $\langle\Phi(m_v+v_0)\delta u_i\rangle = 0$, whereas the last term in (25) is assumed to be 0 because we keep only the terms up to second order in deviations.

As for the fifth term in (25), one arrives at $\left\langle\Phi'(m_v+v_0)\delta u_i\delta v_i\right\rangle = \Phi'(m_v+v_0)\langle\delta u_i\delta v_i\rangle = \Phi'(m_v+v_0)\langle(u_i-m_u)(v_i-m_v)\rangle \equiv \Phi'(m_v+v_0)U$.

Hence, continuing from (25):

$$\langle u_i\Phi(v_i+v_0)\rangle = m_u\Phi(m_v+v_0)$$
$$+\frac{1}{2}m_u\Phi''(m_v+v_0)S_v + \Phi'(m_v+v_0)U \quad (26)$$

As for $\frac{C}{N}\left\langle u_i\sum_{j\in J}\left(u_{i+j}(t-\tau)-u_i\right)\right\rangle$, one may write:

$$\frac{C}{N}\sum_{j\in J}\left\langle u_i(t)u_{i+j}(t-\tau)\right\rangle - \frac{C}{N}\sum_{j\in J}\left\langle u_i^2\right\rangle$$
$$= \frac{C}{N}\sum_{j\in J}\langle u_i\rangle\left\langle u_{i+j}(t-\tau)\right\rangle - \frac{2KC}{N}\left(S_u+m_u^2\right)$$
$$= \frac{2KC}{N}m_u m_u(t-\tau) - \frac{2KC}{N}\left(S_u+m_u^2\right). \quad (27)$$

Finally, proceeding from (24) and incorporating (26) and (27), one obtains:

$$\dot{U} = S_v - S_u\left(1+\frac{2KC}{N}\right) + \Phi'(m_v+v_0)U \quad (28)$$

Summarizing all the results so far, the mean-field model for collective dynamics of N blocks with $2K$ nearest-neighbor interactions reads:

$$\dot{m_u} = m_v$$
$$\dot{m_v} = -m_u - \Phi(v_0) + \Phi(m_v + v_0) + \frac{1}{2}\Phi''(m_v + v_0) S_v + \frac{2KC}{N}(m_u(t-\tau) - m_u)$$
$$\dot{S_u} = 2U$$
$$\dot{S_v} = -2U\left(1 + \frac{2KC}{N}\right) + 2\Phi'(m_v + v_0) S_v + 2D$$
$$\dot{U} = S_v - S_u\left(1 + \frac{2KC}{N}\right) + \Phi'(m_v + v_0) U \quad (29)$$

In order for these equations to describe a self-consistent system, it is required that the values of variances at the stationary state are nonnegative. This is clearly fulfilled given that $S_v^{stac} = -\frac{D}{\Phi'(m_v+v_0)}|_{m_v=0} = \frac{Dv_0}{a} \geq 0$ always holds and $S_u^{stac} = \frac{S_v^{stac}}{1+\frac{2KC}{N}}$.

System (29) could be further simplified by taking into account only the mean values of displacement and velocity, i.e., by assuming that $\dot{S_u}(t) = S_v(t) = \dot{U}(t) = 0$. Such an assumption is justified considering very small deviations from the mean displacement and velocity. In other words, changes of mean displacement and velocity are one order of magnitude higher than changes of corresponding variances and covariance. Hence, the variances and the covariance can be replaced by their stationary values. After this, we finally arrive at the following equations for the mean-field approximated model:

$$\dot{m_u}(t) = m_v(t)$$
$$\dot{m_v}(t) = -m_u(t) + a\ln(v_0) - a\ln(m_v + v_0) + \frac{1}{2}\frac{D}{(m_v + v_0)} + \frac{2KC}{N}(m_u(t-\tau) - m_u(t)) \quad (30)$$

The final form of (30) with included rate-dependent friction term is given in the main text as system (3), whereby the friction law is specified by Eq. (2).

References

1. Burridge, R., Knopoff, L.: Model and theoretical seismicity. Bull. Seismol. Soc. Am. **57**, 341–371 (1967)
2. Dieterich, J.H.: Modeling of rock friction: 1. Experimental results and constitutive equations. J. Geophys. Res. **84**, 2161–2168 (1979)
3. Ruina, A.: Slip instability and state variable friction laws. J. Geophys. Res. **88**(B12), 10359–10370 (1983)
4. Carlson, J., Langer, J.: Mechanical model of an earthquake fault. Phys. Rev. A **40**, 6470–6484 (1989)
5. Kawamura, H., Hatano, T., Kato, N., Biswas, S., Chakrabarti, B.K.: Statistical physics of fracture, friction, and earthquakes. Rev. Mod. Phys. **84**(2), 839–884 (2012)
6. De Sousa Vieira, M.: Chaos and synchronized chaos in an earthquake model. Phys. Rev. Lett. **82**, 201–204 (1999)
7. Erickson, B., Birnir, B., Lavallee, D.: A model for aperiodicity in earthquakes. Nonlinear Process. Geophys. **15**, 1–12 (2008)
8. Kostić, S., Franović, I., Todorović, K., Vasović, N.: Friction memory effect in complex dynamics of earthquake model. Nonlinear Dyn. **73**(3), 1933–1943 (2013)
9. Kostić, S., Franović, I., Perc, M., Vasović, N., Todorović, K.: Triggered dynamics in a model of different fault creep regimes. Sci. Rep. **4**, 5401 (2014)
10. Kostić, S., Vasović, N., Franović, I., Todorović, K.: Dynamics of simple earthquake model with time delay and variation of friction strength. Nonlinear Process. Geophys. **20**, 857–865 (2013)
11. Erickson, B.A., Birnir, B., Lavallée, D.: Periodicity, chaos and localization in a Burridge–Knopoff model of an earthquake with rate-and-state friction. Geophys. J. Int. **187**, 178–198 (2011)
12. Goudarzi, A., Riahi, M.A.: Seismic coherent and random noise attenuation using the undecimated discrete wavelet transform method with WDGA technique. J. Geophys. Eng. **9**, 619–631 (2012)
13. Chiu, S.K.: Coherent and random noise attenuation via multichannel singular spectrum analysis in the randomized domain. Geophys. Prospect. **61**, 1–9 (2013)
14. Pomeau, Y., Le Berre, M.: Critical speed-up versus critical slow-down: a new kind of relaxation oscillation with application to stick-slip phenomena. arXiv:1107.3331v1 (2011)
15. Mori, T., Kawamura, H.: Simulation study of spatiotemporal correlations of earthquakes as a stick-slip frictional instability. Phys. Rev. Lett. **94**, 058501-1-4 (2005)
16. Xia, J., Gould, H., Klein, W., Rundle, J.B.: Near-mean-field behavior in the generalized Burridge-Knopoff earthquake model with variable-range stress transfer. Phys. Rev. E **77**, 031132-1-11 (2008)
17. Mori, T., Kawamura, H.: Simulation study of earthquakes based on the two-dimensional Burridge–Knopoff model with long–range interactions. Phys. Rev. E **77**, 051123-1-16 (2008)
18. Varotsos, P.A., Sarlis, N.V., Skordas, E.S., Uyeda, S., Kamogawa, M.: Natural-time analysis of critical phenomena: the case of seismicity. Europhys. Lett. **92**, 29002 (2010)
19. Lindner, B., Garcia-Ojalvo, J., Neiman, A., Schimansky-Geier, L.: Effects of noise in excitable systems. Phys. Rep. **392**, 321 (2004)
20. Zaks, M.A., Sailer, X., Schimansky-Geier, L., Neiman, A.B.: Noise induced complexity: from subthreshold oscillations to spiking in coupled excitable systems. Chaos **15**, 026117 (2005)
21. Klinshov, V., Franović, I.: Mean-field dynamics of a random neural network with noise. Phys. Rev. E
22. Telford, W.M., Geldart, L.P., Sheriff, R.E.: Applied Geophysics, 2nd edn. Cambridge University Press, Cambridge (1990)

23. Ryabov, V.B., Correig, A.M., Urquizu, M., Zaikin, A.A.: Microseism oscillations: from deterministic to noise-driven models. Chaos Solitons Fract. **16**, 195–210 (2003)
24. Sone, H., Shimamoto, T.: Frictional resistance of faults during accelerating and decelerating earthquake slip. Nat. Geosci. **2**, 705–708 (2009)
25. Lapusta, N.: The roller coaster of fault friction. Nat. Geosci. **2**, 676–677 (2009)
26. Burić, N., Ranković, D., Todorović, K., Vasović, N.: Mean field approximation for noisy delay coupled excitable neurons. Phys. A **389**, 3956–3964 (2010)
27. Vasović, N., Kostić, S., Franović, I., Todorović, K.: Earthquake nucleation in a stochastic fault model of globally coupled units with interaction delays. Commun. Nonlinear Sci. **38**, 117–129 (2016)
28. Shimazaki, K., Nakata, T.: Time-predictable recurrence model for large earthquakes. Geophys. Res. Lett. **7**, 279–282 (1980)
29. Kanamori, H.: Earthquake prediction: an overview. Int. Geophys. **81**(part B), 1205–1216 (2003)
30. Parsons, T., Console, R., Falcone, G., Muru, M., Yamashina, K.: Comparison of characteristic and Gutenberg–Richter models for time-dependent $M \geq 7.9$ earthquake probability in the Nankai–Tokai subduction zone, Japan. Geophys. J. Int. **190**(3), 1673–1688 (2012)