

## Angular Momentum and Topological Dependence of Kepler's Third Law in the Broucke-Hadjidemetriou-Hénon Family of Periodic Three-Body Orbits

Marija R. Janković

*Faculty of Physics, Belgrade University, Studentski Trg 12, 11000 Belgrade, Serbia*

V. Dmitrašinović

*Institute of Physics, Belgrade University, Pregrevica 118, Zemun, P.O. Box 57, 11080 Belgrade, Serbia*

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We use 57 recently found topological satellites of Broucke-Hadjidemetriou-Hénon's periodic orbits with values of the topological exponent  $k$  ranging from  $k = 3$  to  $k = 58$  to plot the angular momentum  $L$  as a function of the period  $T$ , with both  $L$  and  $T$  rescaled to energy  $E = -0.5$ . Upon plotting  $L(T/k)$  we find that all our solutions fall on a curve that is virtually indiscernible by the naked eye from the  $L(T)$  curve for nonsatellite solutions. The standard deviation of the satellite data from the sixth-order polynomial fit to the progenitor data is  $\sigma = 0.13$ . This regularity supports Hénon's 1976 conjecture that the linearly stable Broucke-Hadjidemetriou-Hénon orbits are also perpetually, or Kol'mogorov-Arnol'd-Moser, stable.

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*Introduction.*—Numerical studies of periodic three-body orbits have increased their output over the past few years—more than 40 new orbits—and their “satellites” have been discovered, Refs. [1–4]. Unlike periodic two-body orbits, which are all ellipses, and thus are all topologically equivalent, the noncolliding three-body periodic orbits have one of infinitely many different topologies. Montgomery, Ref. [5], had devised an algebraic method to associate a free-group element (“word”)  $w$  with a three-body orbit's topology, and thus to label and classify such periodic orbits; for an elementary introduction to this method, see Ref. [6]. That classification method has recently acquired practical importance in the identification of new three-body orbits, Refs. [1,3,4].

A number of newly discovered orbits, Refs. [1–4], were of the so-called topological satellite type. Such satellite orbits are also known as “bifurcation” in the older literature, Refs. [2,7], where they were only loosely defined in terms of their presumed origin. It was only in Ref. [3] that a precise definition of a topological satellite was given. When this definition was applied to the figure-eight satellites [8], reported in Ref. [3], it led to the discovery of a remarkable “topological Kepler's third law”-like regularity for arbitrary orbits with vanishing angular momenta, Ref. [9]. The immediate question is whether this regularity persists when the angular momentum does not vanish.

The present Letter is an attempt to answer that question, albeit in a single, specific family of three-body orbits, viz., in the Broucke-Hadjidemetriou-Hénon (BHH) family [10–16], that has the simplest nontrivial topology (free group element  $w = a$ ). The main reason for selecting only this family of orbits is that it is the most thoroughly studied family thus far: it is the only family of orbits with a previously determined dependence of the period  $T$  on the

angular momentum  $L$  of (nonsatellite, or progenitor) periodic orbits, Refs. [10–16]. No such, or comparable, study of any of the remaining known families exists to our knowledge at this moment. Moreover, the BHH family is one of only two families [17] of periodic three-body orbits that have been observed in astronomy: all known “hierarchical” triple star systems belong to BHH orbits. Moreover, the Sun-Earth-Moon system may be viewed as a BHH solution, albeit with highly asymmetrical mass ratios.

The first step towards this goal, the one of finding as many different BHH satellite orbits as possible, has already been accomplished in Ref. [18]. Previously, Davoust and Broucke, Ref. [7], had found one (the first  $k = 3$ ) satellite of one retrograde BHH orbit. Reference [18] extended the search for retrograde BHH satellite orbits systematically up to values  $k \leq 19$  of the topological exponent  $k$ , and more haphazardly up to  $k = 58$ . Thus, several different types of BHH satellites with identical values of  $k$  were discovered [19], as were a few prograde BHH satellites; see the Supplemental Material [20] and the Web site [21]. Prograde BHH satellites have not been studied systematically, as yet, mostly due to their paucity at the values of the angular momentum covered in the searches in Ref. [18]. Presently, it is not known how many satellites ought to exist, and under which conditions. It is interesting, however, that the observed satellites correspond only to linearly stable BHH progenitor orbits. This is in line with Hénon's 1976 conjecture [15,20] about Kol'mogorov-Arnol'd-Moser (KAM) stability of linearly stable BHH orbits.

Then, motivated by the findings reported in Ref. [9], we checked for similar regularities of satellite BHH orbits with nonzero angular momentum. First, we formulated the topological dependence of Kepler's third law for three-body orbits with nonzero angular momenta, and second, we

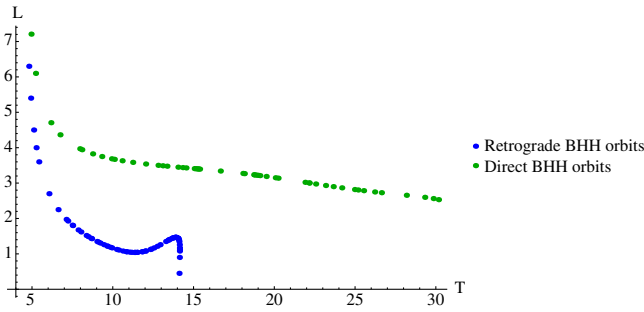


FIG. 1.  $L(T)$  curves for direct or prograde (green, upper set of points) and retrograde (blue, lower set of points) BHH orbits, all at fixed energy  $E = -0.5$ .

tested it on the presently known satellites of the retrograde BHH family. We found a striking result: all of our retrograde BHH satellites fall on a single (continuous) curve  $L(T/k)$ , Fig. 3, that is practically indiscernible by the naked eye from the  $L(T)$  curve, Fig. 1, for nonsatellite (progenitor) retrograde BHH solutions, whereas the “topologically uncorrected” curve  $L(T)$  looks very different; see Fig. 2. A quantitative measure of this (dis)agreement is shown in terms of corresponding standard deviations.

*Preliminaries.*—Broucke [7,10,11], Hadjidemetriou [12–14], and Hénon [15,16] (BHH) explored a set of periodic planar three-body orbits with equal mass bodies. These orbits form two continuous curves in the  $L$ - $T$  plane whose lower (retrograde) terminus (“end”) is the collinear collision (Schubart) orbit, and both the retrograde and the direct  $L(T)$  curves approach the same high- $L$  limit at their upper termini, Fig. 1.

Although BHH write of two families of orbits—direct, or prograde, and retrograde—all of these orbits belong to a single topological family: during one period the orbit completes a single “loop” around one of the poles on the shape sphere. This loop can be described by the conjugacy class of the fundamental group or free group element  $a$ , according to the topological classification used in Refs. [1,6]. It turns out, however, that there are numerous relative periodic orbits with topology  $a^k$ , with  $k = 2, 3, \dots$

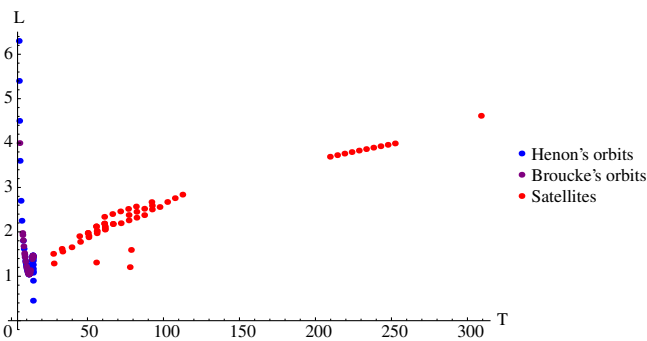


FIG. 2.  $L(T)$  dependence of retrograde BHH orbits (blue dots of different hues) and their satellites (red), with various values of  $k$ , all at fixed energy  $E = -0.5$ . The data are from Table I.

Such orbits are sometimes called satellites [2,3], whereas other authors call them “bifurcation orbits” [7].

*Scaling laws for three bodies.*—It is well known that Kepler’s third law (for two bodies) follows from the spatiotemporal scaling laws, which, in turn, follow from the homogeneity of the Newtonian gravity’s static potential, Ref. [22]. These scaling laws read  $\mathbf{r} \rightarrow \lambda \mathbf{r}$ ,  $t \rightarrow \lambda^{3/2} t$ , and, consequently,  $\mathbf{v} \rightarrow \mathbf{v}/\sqrt{\lambda}$ . The (total) energy scales as  $E \rightarrow \lambda^{-1} E$ , the period  $T$  as  $T \rightarrow \lambda^{3/2} T$ , and angular momentum as  $L \rightarrow \lambda^{1/2} L$ , i.e., differently than either the period  $T$ , or “size”  $R$ , which is the reason why only the vanishing angular momentum  $L = 0$  is a “fixed point” under scaling. For this reason, we use scale-invariant angular momentum  $L_r = L|E|^{1/2}$ , scale-invariant period  $T_r = T|E|^{3/2}$  and, for simplicity’s sake, equal masses. Thus, we may replace the “mean size”  $\bar{R}$  of the three-body system in Kepler’s third law  $T \propto \bar{R}^{3/2}$  with the inverse absolute value of energy  $|E|^{-1}$ , i.e.,  $T \propto |E|^{-3/2}$ , or equivalently  $T|E|^{3/2} = T_r = \text{const}$ .

The “constant” on the right-hand side of this equation is not a universal one in the three-body case, as it is in the two-body case (where it depends only on the masses and the Newtonian coupling  $G$ ). It may depend both on the family  $w$  of the three-body orbit, described by the free-group word  $w$ , and on the scale-invariant angular momentum  $L_r = L|E|^{1/2}$  of the orbit, see Refs. [15,16], as follows:

$$T^{(w)}|E|^{3/2} = T_r^{(w)} = f(L^{(w)}|E|^{1/2}) = f(L_r^{(w)}),$$

or as an inverse function,

$$L_r^{(w)} = L^{(w)}|E|^{1/2} = f^{-1}(T^{(w)}|E|^{3/2}) = f^{-1}(T_r^{(w)}).$$

Thus, the curve  $L_r^{(w)}(T_r^{(w)}) = L^{(w)}|E|^{1/2}(T^{(w)}|E|^{3/2})$  as a function of  $T_r^{(w)} = T^{(w)}|E|^{3/2}$  is a fundamental property of any family  $w$  of periodic orbits. For the BHH family the  $L(T)$  curve, for fixed energy  $E = -0.5$  orbits, based on the data from Refs. [10–16], is shown in Fig. 1.

We wish to see if the zero-angular-momentum relation  $T_r(w^k) = kT_r(w)$ , Ref. [9], or some similar statement holds also at nonzero angular momentum. The analogon of this relation for orbits with nonzero angular momenta would be a simple relation between  $L(T)$  curves for the progenitor orbit  $L_r(T_r)$  and its  $k$ th satellite  $L_r^{(w^k)}(T_r^{(w^k)})$ :

$$L_r^{(w)}(T_r^{(w)}) = L_r^{(w^k)}(T_r^{(w^k)})/k. \quad (1)$$

We shall test this relation in the BHH family of solutions, and in order to do so, we use the BHH satellite orbits from Ref. [18].

*$L(T)$  curves for BHH satellites.*—The  $L$ - $T$  plots of different- $k$  satellite orbits are scattered over a large region and do not intersect the BHH progenitor family of orbits’

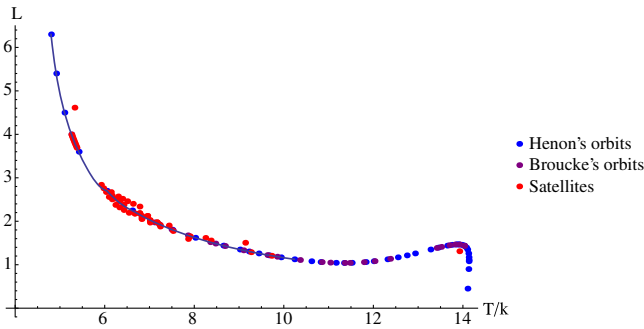


FIG. 3.  $L(T' = T/k)$  dependence at fixed energy  $E = -0.5$  for the aggregate set of retrograde BHH orbits (blue dots of different hues) and their satellites (red dots) with various values of  $k$ , together with the fitted interpolating function (blue solid). The data are from Table I.

$L(T)$  curve when plotted as a function of the (undivided) period  $T$ , see Fig. 2. Note the large span of periods  $T$  in the data, Table I, and in Fig. 2, as well as two large “gaps” in the data. These gaps are due to the exigencies of the search reported in Ref. [18], which was not conducted with the intention of testing the hypothetical topological Kepler’s third law. The values in Table I have been rounded off to five significant decimal places. So, the numerical error is less than one part in 10 000. Such an error would be invisible in Figs. 2, 3, and 4, meaning that the “size of the points” in these figures is larger than the expected error. After dividing the period  $T$  (at fixed energy) by the topological exponent  $k$ ,  $T' = T/k$ , we can see in Fig. 3 that the satellite orbits’  $L(T/k)$  curve (the angular momentum  $L$  as a function of topologically rescaled period  $T/k$ ) approximately coincides with the  $L(T)$  curve of BHH retrograde orbits. It seems that such an appearance of order out of apparent disorder cannot be an accident.

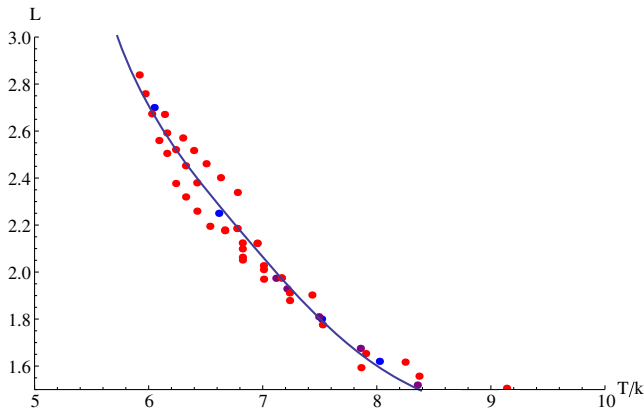


FIG. 4. Enlargement of the  $L \in [1.5, 3]$  region of the retrograde BHH orbits (blue dots) and their satellites (red dots) with various values of  $k$ .  $L(T' = T/k)$  dependence at fixed energy  $E = -0.5$ . Note that the size of the dots on the diagram exceeds the corresponding numerical uncertainties (“error bars”).

Next, in Fig. 3 we look more closely at the section of the  $L(T)$  curve of progenitor BHH retrograde orbits in which we have found all but one of our satellites. We have interpolated Hénon’s [15] 18 stable retrograde data points with a piecewise polynomial fit in this part of the  $L(T)$  curve. The standard deviations from this interpolated curve were calculated for (1) Broucke’s 10 progenitor retrograde orbits [10,11] and (2) the 56 out of 57 new satellite orbits from Table I (excluding one orbit that lies near the “shoulder” at  $T = 14$  in Fig. 3), with the following results. (1)  $\sigma = 0.0034$  for Broucke’s orbits, and (2)  $\sigma = 0.1269$  for satellite orbits. This difference of 2 orders of magnitude between these two numbers clearly indicates that the rescaled satellites’ periods do *not* coincide *exactly* with the progenitor ones, but only approximately.

Moreover, when one assembles Hénon’s and Broucke’s [10,11] retrograde orbits in one set and fits the aggregate data by a polynomial of the sixth degree, Fig. 3, the standard deviation of the fit is  $\sigma = 0.0313$ , whereas the

TABLE I. Properties of satellite orbits in the retrograde branch of the BHH family. Here,  $k$  is the topological power of the orbit,  $T$  is its period, and  $L$  is its angular momentum. All orbits have the same energy  $E = -(1/2)$ . For the raw data and a discussion of numerical errors, see the Supplemental Material [20].

$T$	$L$	$k$	$T$	$L$	$k$
27.800 80	1.288 15	3	71.538 38	2.460 95	11
27.411 57	1.505 52	3	77.074 74	2.259 18	12
32.992 45	1.616 82	4	77.060 60	2.379 81	12
33.479 35	1.557 01	4	76.731 11	2.517 18	12
55.678 84	1.310 00	4	82.213 27	2.319 68	13
39.511 02	1.653 31	5	82.199 18	2.452 31	13
45.138 27	1.775 68	6	81.882 58	2.570 68	13
44.586 32	1.902 40	6	87.317 60	2.376 87	14
50.646 60	1.879 00	7	87.303 60	2.520 98	14
50.638 90	1.911 39	7	92.384 79	2.504 86	15
50.141 13	1.974 52	7	92.377 38	2.591 66	15
50.141 28	1.975 37	7	92.082 10	2.670 70	15
56.060 83	1.969 71	8	97.432 10	2.559 79	16
55.604 11	2.121 89	8	102.450 58	2.673 31	17
77.813 66	1.205 44	8	107.449 64	2.758 61	18
56.052 69	2.010 54	8	112.429 18	2.838 83	19
56.049 53	2.027 09	8	209.487 95	3.692 20	39
55.604 30	2.122 89	8	214.258 15	3.727 85	40
61.399 03	2.051 28	9	219.023 02	3.762 83	41
60.968 89	2.185 81	9	223.782 78	3.797 19	42
61.390 86	2.098 90	9	228.537 63	3.830 94	43
61.386 76	2.123 97	9	233.287 75	3.864 12	44
60.968 79	2.185 32	9	238.033 32	3.896 75	45
60.999 96	2.338 82	9	242.774 50	3.928 85	46
61.396 97	2.063 00	9	247.511 46	3.960 44	47
66.666 44	2.179 17	10	252.244 33	3.991 55	48
66.666 89	2.176 08	10	308.853 30	4.614 04	58
66.297 61	2.401 65	10			
78.610 58	1.593 25	10			
71.897 15	2.194 81	11			

standard deviation of all satellite orbits from this polynomial curve is  $\sigma = 0.1315$ , roughly four times bigger. It is (statistically) clear that the satellites do not follow exactly the same  $L(T)$  curve as the progenitors, but the deviation is not large. This constitutes the evidence for the analogon of the topological dependence of Kepler's third law for the  $L \neq 0$  case, Ref. [9].

Finally, we note that all of our newly found satellite orbits fall into a region of the progenitor  $L(T)$  curve that corresponds to stable progenitor BHH orbits, with one possible exception (the red point near the shoulder at  $T = 14$  in Fig. 3, that “sits” on the border point between stable and unstable regions). We have not found any other satellites in this, the second stable region of BHH retrograde orbits. In Fig. 4 we show the fine structure in the satellites'  $L(T/k)$  curve, that remains to be studied in finer detail and be better understood.

We have not studied the direct or prograde (sub)family of BHH orbits, as Ref. [18] did not search for their satellites, but found four almost inadvertently. Certainly, that task ought to be completed in the future.

*Summary, conclusions and outlook.*—We have used 57 new satellite orbits from Ref. [18], in the family of Broucke-Hadjidemetriou-Hénon, Refs. [10–16], relative periodic solutions to the planar three body problem. Thence followed a striking relation between their kinematic and topological properties.

BHH orbits constitute a family with a simple topology, described by the free group element  $a$  according to the classification on the shape sphere, and their satellites are orbits of the topology  $a^k$ . The BHH orbits' angular momenta  $L$  and periods  $T$  form a continuous curve  $L(T)$ , at fixed energy. Our satellite orbits form a scattered set of points on the same  $L(T)$  plot, but all of them exhibit the property that after their period  $T$  is divided by their topological order  $k$ , they approximately fall on the  $L(T)$  curve of the original ( $k = 1$ ) BHH orbits.

This study was motivated by the discovery, Ref. [9], of a relation between the topology and periods among the satellites of the figure-eight orbit, Ref. [3], and one other type (“moth  $I$ ”—“yarn” in Ref. [1]), of three-body orbits at vanishing angular momentum. This Letter shows that Kepler's third law's topological dependence also holds for orbits with  $L \neq 0$ , albeit only approximately. It remains to be seen just precisely what this discrepancy depends on.

These results are even more striking if one remembers that among our results there are several distinct types of satellite orbits of the same topological power  $k$ , some with quite different values of  $L$  and  $T$ , which all display this property. A closer look at the  $L(T/k)$  curve revealed a fine structure, which should be investigated in higher detail in the future. An extension of the search conducted in Ref. [18] into hitherto unexplored regions of the  $L-T$  plane ought to provide (new) data that will further test our result.

Our results indirectly confirm Hénon's 1976 conjecture, see page 282 in Ref. [15], reproduced in the Supplemental Material [20], that the linearly stable BHH orbits are also nonlinearly, or perpetually, or KAM stable. Such KAM stability implies the existence of quasiperiodic orbits with periods that conform to the quasiperiodicity condition (i.e., with periods that are “almost commensurate” with the BHH progenitor's period), as predicted by the KAM theorem, Refs. [23–25].

Our study opens several new questions. (1) The most commonly observed hierarchical triple star systems belong to the BHH family. Are there BHH topological satellites among astronomically observed three-body systems? It is important to extend the present study to the realistic case of three different masses: some early work has already been done in this direction by Broucke and Boggs, Ref. [10], and by Hadjidemetriou and Christides, Ref. [13]. (2) In recent years there have been formal “proofs of existence” given for at least some BHH orbits, Refs. [26,27]. This begs the question: can one “prove existence” of their satellite orbits, and, if yes, of how many satellites, and under which conditions?

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- [20] See the Supplemental Material at <http://link.aps.org/supplemental/10.1103/PhysRevLett.116.064301> for The search method is described, tables with the data regarding the new BHH satellites, together with the numerical uncertainties, are shown. We briefly discuss Hénon's conjecture about KAM stability of BHH orbits.
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