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Research Article

p-Wave Pairing in Quantum Hall Bilayers

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We show that the wave functions that describe the ground states of putative p-wave-paired phases in quantum Hall bilayers, like the Pfaffian at $\nu_T = 1/2$ or the paired phase at $\nu_T = 1$, are more likely to describe the excited states of Fermi liquids at these filling factors. We point out to the close competition between Fermi liquid and paired phases, which leads to the conclusion that in the experiments only direct transitions from the correlated 111 and 331 states into Fermi liquid(s) are likely to be observed.

1. Introduction

The quantum Hall bilayer, which consists of two layers of 2D electron gases in the presence of strong magnetic field orthogonal to the layers, represents a stage for correlated states with an extra layer degree of freedom [1]. When the total filling factor is $\nu_T = 1/2$ and the distance d between the layers is on the order of the magnetic length, $d \sim l_B$, we expect the so-called 331 state, which can be viewed as a Cauchy determinant pairing among the underlying quasiparticles which are neutral fermions. This is a p-wave pairing between two species of neutral fermions belonging to different layers. The pairing or order parameter function should be an eigenstate of rotations in two dimensions and, in the p-wave case, assumes the form [2] $\Delta_{\mathbf{k}} = \Delta(k_x - ik_y)$, where $\mathbf{k}$ is the relative momentum of the Cooper pair and Δ a constant. Antisymmetrizing a Cauchy determinant leads to the Pfaffian wave function [3]. The physical question therefore is whether the tunneling in the quantum Hall bilayer can mimic the antisymmetrization between the two species of electrons. Tunneling would allow to reach a p-wave-paired state of electrons without the layer degree of freedom, that is, the Moore-Read state. Recent work [4] pointed out that this route to the Pfaffian may be obstructed by the intervening phase of the Fermi liquid of neutral fermions [5]. The latter is, for all that we know, the only stable phase of one-component neutral fermions at the filling factor $\nu = 1/2$ in the lowest Landau level (LLL) and would constitute a certain outcome of strong tunneling. On the other hand, for the quantum Hall bilayer at total filling factor $\nu_T = 1$ and for small distances we have a highly correlated excitonic 111 state [6, 7]. For large distances between the two layers, two separate Fermi liquids of neutral fermions are formed, each at the filling factor 1/2. Proposals [8–12] were made for an intervening p-wave-paired phase between electrons—neutral fermions, that belong to different layers at intermediate distances.

In this paper, by analytical and numerical means, we further address the relevance of the proposals for the p-wave phases: (a) Pfaffian in the context of bilayer at $\nu_T = 1/2$ and (b) the BCS-like (with two species) p-wave wave function for the bilayer at $\nu_T = 1$. In both cases, we assume the problem is defined by the usual Coulomb interaction. We point out to the close competition between Fermi liquid(s) and paired phases in these systems, suggesting that direct transitions from the correlated 111 and 331 states into Fermi liquid(s) are likely to take place in the experiments.

2. The Quantum Hall Bilayer and Its Model Wave Functions

In the following, we review some basic facts about the quantum Hall bilayer including its possible phases and the corresponding quantum Hall wave functions. We consider the quantum Hall bilayer in the presence of the vector

potential $\mathbf{A}$ that describes a strong magnetic field, $B\hat{z} = \nabla \times \mathbf{A}$, perpendicular to both layers. In the rotationally symmetric gauge, the lowest Landau level (LLL) eigenstates of an electron with the coordinate $z = x + iy$ in the plane and localized in the layer $\sigma \in \{\uparrow, \downarrow\}$ are given by

$$z^m \exp\left\{-\frac{|z|^2}{4l_B^2}\right\}\eta_\sigma, \quad m = 0, \ldots, N_\phi - 1, \tag{1}$$

where η_σ is the usual spinor wave function and the unit of length is given by the magnetic length, $l_B = \sqrt{\hbar c/eB}$. The number of flux quanta, N_ϕ, denotes the number of available states in the LLL. In the thermodynamic limit, the ratio of the total number of electrons N_e and the number of flux quanta N_ϕ defines the filling factor $\nu_T = N_e/N_\phi$. We focus on the filling factors $\nu_T = 1/2$ and $\nu_T = 1$.

The many-body interacting system of electrons is defined by the following Lagrangian density in the second quantized formulation:

$$\begin{aligned}\mathcal{L} = \sum_\sigma \Bigg\{ & \Psi_\sigma^\dagger \partial_\tau \Psi_\sigma - \Psi_\sigma^\dagger \frac{(\partial_{\mathbf{r}} + e\mathbf{A})^2}{2m}\Psi_\sigma - \Psi_\sigma^\dagger \frac{\Delta_{\mathrm{SAS}}}{2}\Psi_{-\sigma} \\ & + \frac{1}{2}\int d\mathbf{r}' \rho_\sigma(\mathbf{r}) V_c^{\mathrm{intra}}(\mathbf{r}-\mathbf{r}')\rho_\sigma(\mathbf{r}') \\ & + \frac{1}{2}\int d\mathbf{r}' \rho_\sigma(\mathbf{r}) V_c^{\mathrm{inter}}(\mathbf{r}-\mathbf{r}')\rho_{-\sigma}(\mathbf{r}') \Bigg\},\end{aligned} \tag{2}$$

where Ψ_σ is the electron field which carries the pseudospin (layer) index and Δ_{SAS} denotes the tunneling term. The interaction is defined by

$$V_c^{\mathrm{intra}}(r) = \frac{e^2}{\epsilon r} \tag{3}$$

and in general V_c^{inter} is different. When we model a quantum Hall bilayer,

$$V_c^{\mathrm{inter}}(r) = \frac{e^2}{\epsilon\sqrt{r^2 + d^2}}, \tag{4}$$

d has the meaning of distance between the two layers of 2D gases and it is of the order of l_B. (In the Lagrangian density (2) we set $\hbar = c = l_B = 1$.) Significant insight into the physics described by the Lagrangian (2) can be obtained using first-quantized trial wave functions for its ground states [13]. We now list several candidate wave functions that are expected to describe the ground state of (2) in different limits of Δ_{SAS} and d for the filling factors $\nu_T = 1/2$ and $\nu_T = 1$. Trial wave functions in the LLL are analytic in z variables and we will omit the omnipresent Gaussian factor for each electron as the one in (1).

In the small tunneling regime, the fractional quantum Hall (FQH) system at $\nu_T = 1/2$ is two-component, described by the 331 Halperin state for two distinguishable species of electrons, $z_{i\sigma}; \sigma = \uparrow, \downarrow;\ i = 1, \ldots, N_e/2$,

$$\Psi_{331} = \prod_{i<j}\left(z_{i\uparrow} - z_{j\uparrow}\right)^3 \prod_{k<l}(z_{k\downarrow} - z_{l\downarrow})^3 \prod_{p,q}\left(z_{p\uparrow} - z_{q\downarrow}\right). \tag{5}$$

Due to the fact that the correlation exponents between electrons of the same layer are bigger than those between electrons of the opposite layers, we expect the wave function (5) to be more appropriate for nonzero d, for example, in the range $d \sim l_B$.

As the tunneling strength Δ_{SAS} is increased, the electrons find it energetically favorable to be in the superposition of two layers, $\uparrow + \downarrow$, and the system loses its two-component character. The effective single-component state is characterized by full polarization in the x-direction. At $\nu = 1/2$ single layer in the LLL, a compelling candidate for the polarized state is Rezayi-Read composite Fermi liquid state [14]:

$$\Psi_{1/2} = \mathcal{P}_{\mathrm{LLL}}\left\{\mathcal{F}(\{z, \bar{z}\}) \cdot \prod_{i<j}\left(z_i - z_j\right)^2\right\}, \tag{6}$$

where $\mathcal{P}_{\mathrm{LLL}}$ is a projector to the LLL and $\mathcal{F} \equiv \det[e^{i\mathbf{k}_i \mathbf{r}_j}]$ represents the Slater determinant of free waves. Note that a single index suffices to label the electron coordinates as the pseudospin index is implicitly assumed to be $\uparrow + \downarrow$.

An alternative candidate for the polarized state at the half filling is the so-called Pfaffian state, which up to a normalization factor can be expressed as

$$\Psi_{\mathrm{Pf}} = \mathcal{A}\{\Psi_{331}\}. \tag{7}$$

The two-component state is made single-component under the action of the antisymmetrizer $\mathcal{A}$ between $\uparrow$ and $\downarrow$ electron coordinates. In the notation of (7), one can think of the Pfaffian originating from the two-component 331 state with the pairing represented by the Cauchy determinant because

$$\Psi_{331} \propto \Psi_{222} \det\left[\frac{1}{z_\uparrow - z_\downarrow}\right], \tag{8}$$

by virtue of the Cauchy identity. As a result, we can express the Pfaffian state in its more familiar form as

$$\begin{aligned}\Psi_{\mathrm{Pf}} \propto & \prod_{i<j}\left(z_i - z_j\right)^2 \\ & \times \sum_{\sigma \in S_n} \operatorname{sgn}\sigma \left\{\frac{1}{(z_{\sigma(1)} - z_{\sigma(2)})} \cdots \frac{1}{(z_{\sigma(N_e-1)} - z_{\sigma(N_e)})}\right\},\end{aligned} \tag{9}$$

where the sum represents a BCS wave function for spinless fermions that pair in the manner of a p-wave (see below). Pfaffian (7) is usually thought of as a candidate wave function to describe the half filling of the *second* Landau level, where the nature of the effective interaction facilitates the pairing between the electrons [15]. However, being an analytic wave function, Pfaffian is also a valid trial state for the LLL.

At $\Delta_{\mathrm{SAS}} = 0$ and in the small d regime, the FQH system at $\nu_T = 1$ is two-component, described by the 111 Halperin state for two distinguishable species of electrons,

$$\Psi_{111} = \prod_{i<j}\left(z_{i\uparrow} - z_{j\uparrow}\right)\prod_{k<l}(z_{k\downarrow} - z_{l\downarrow})\prod_{p,q}\left(z_{p\uparrow} - z_{q\downarrow}\right). \tag{10}$$

The correlation exponents are the same between the electrons belonging to the same and opposite layers, reflecting the excitonic nature [6] of the correlated state in the regime when the inter- and intrainteractions are about the same. As the distance between the layers is increased, it is expected that electrons find it energetically more favorable to correlate inside the layers and make two independent composite Fermi liquid states, each in the form of (6).

When we study the trial states on a spherical surface, we will make use of the fact that they are characterized by the topological number called the shift, $\mathcal{S} = N_e/\nu - N_\phi$. In the case of the 331 state $\mathcal{S} = 3$, for the 111 state $\mathcal{S} = 1$, and for the Fermi liquid state in (6) $\mathcal{S} = 2$. The shift is a distinctive feature of each FQH state on the sphere.

3. The Role of p-Wave Pairing for the Transitions from 331 and 111 State into Fermi Liquids

We start with the effective description of [16] of the p-wave-paired state in order to describe its instability towards the Fermi liquid. We will show how the paired wave function, from representing a ground state in the paired phase, ends up describing an excited state of the Fermi liquid after the transition. The BCS effective description near $\mathbf{k} = 0$ is given by

$$K_{\text{eff}} = \sum_{\mathbf{k}} \left\{ (\epsilon_{\mathbf{k}} - \mu) c_{\mathbf{k}}^{\dagger} c_{\mathbf{k}} + \left(\Delta_{\mathbf{k}} c_{\mathbf{k}}^{\dagger} c_{-\mathbf{k}}^{\dagger} + h.c. \right) \right\}, \tag{11}$$

where $\Delta_{\mathbf{k}} = \Delta(k_x - ik_y)$ and the diagonal term describes fermions with a quadratic dispersion in 2D. After the Bogoliubov transformation with operators in the form $\alpha_{\mathbf{k}} = u_{\mathbf{k}} c_{\mathbf{k}} - v_{\mathbf{k}} c_{-\mathbf{k}}^{\dagger}$, we obtain for the dispersion of the excitations above the ground state

$$E_{\mathbf{k}} = \sqrt{(\epsilon_{\mathbf{k}} - \mu)^2 + |\Delta_{\mathbf{k}}|^2}, \tag{12}$$

and the ground state is

$$\prod_{\mathbf{k}} \left(1 + g_{\mathbf{k}} c_{\mathbf{k}}^{\dagger} c_{-\mathbf{k}}^{\dagger} \right) |0\rangle, \tag{13}$$

where $g_{\mathbf{k}} = v_{\mathbf{k}}/u_{\mathbf{k}}$. For small $\mathbf{k}$ and $\mu > 0$, this becomes

$$\frac{v_{\mathbf{k}}}{u_{\mathbf{k}}} = -\frac{(E_{\mathbf{k}} - \epsilon_{\mathbf{k}} + \mu)}{\Delta_{\mathbf{k}}^{*}} \longrightarrow -\frac{2\mu}{\Delta_{\mathbf{k}}^{*}}, \tag{14}$$

which then in the real space yields asymptotically $g(\mathbf{r}) \sim 1/z$, that is, the p-wave Cooper pair pairing function. The antisymmetrization over a collection of these Cooper pairs leads to the Moore-Read state or the Pfaffian where the pairing is described by $1/z$ for all distances [16].

If $\mu > 0$ becomes large and $\Delta_{\mathbf{k}}$ stays about the same or decreases, then due to the simple expansion for small k the energy of excitations behaves as

$$E_{\mathbf{k}} \approx \mu - \epsilon_{\mathbf{k}} + \frac{|\Delta_{\mathbf{k}}|^2}{2\mu} \tag{15}$$

and the minimum around $\mathbf{k} = 0$ becomes a local maximum. Then we may think of this unstable point in the phase space as a description of an excited state of the system which finds itself, with high probability, in the state of Fermi liquid because the minimum for excitations has moved to $\mathbf{k} = \mathbf{k}_F$ from $\mathbf{k} = 0$. But the description of this unstable point, that is the excited state is still given by (13) and (14), that is, by a Pfaffian pairing function. Therefore Pfaffian may describe an excited state of the Fermi liquid.

In the following, we will discuss the two correlated systems, at $\nu_T = 1/2$ and $\nu_T = 1$, where the p-wave-paired state may occur as an excited state of the Fermi liquid.

331 Case. In the presence of tunneling, the 331 bilayer state can be transformed into a Fermi liquid of neutral or composite fermions in the LLL. This expectation is based on the fact that Pfaffian, as an alternative candidate, is not found in any polarized system (without an internal degree of freedom) in the LLL. BCS theory of the 331 system can be found in [4, 16]. There is a separation of the description into two sectors, the even and odd channel. Each channel is described by the Read-Green theory of the p-wave pairing of spinless fermions that we introduced above. With the increase of tunneling Δ_{SAS}, the number of electrons in the even channel increases and its effective chemical potential is modified as $\mu^e = \mu + \Delta_{\text{SAS}}/2$. The excitations of the even channel, by having μ^e instead of μ above as a parameter that enters the expression for their energies (12), become unstable unless there is an appropriate change in $\Delta_{\mathbf{k}}$, for example, the strengthening of the pairing. In the absence of this change, the system evolves into a Fermi liquid with a possibility that one of the excited states is described by the Pfaffian (13), as argued previously. Exact diagonalization calculations on the sphere, performed at the fixed value of $d = l_B$ and at the fixed shift $\mathcal{S} = 3$, characteristic of the Pfaffian state, give high overlaps for the Pfaffian with the increase of tunneling [4]. On the other hand, exact diagonalization on the torus [4], with the same fixed distance between the layers, does not find the characteristic degeneracy of the Pfaffian with the increase of tunneling [4]. A possible interpretation of the result on the sphere is that the true ground state of the system is found at a different shift $\mathcal{S}$, where the Pfaffian describes, to a high accuracy, an excited state of the system. This is corroborated by comparing the ground state energies on the sphere as a function of tunneling for the shifts $\mathcal{S} = 3$ (Pfaffian) and $\mathcal{S} = 2$ (the shift of the Fermi liquid); see Figure 1. We show the data for two systems of $N = 8$ and $N = 10$ electrons which do not suffer from the aliasing with Jain's composite fermion states [17]. The energies plotted in Figure 1 include the appropriate background charge and finite-size corrections [17]. We observe that the polarized state (for large Δ_{SAS}) is found at the Fermi liquid shift of $\mathcal{S} = 2$ whereas the excited state lying closely above it, at the shift of $\mathcal{S} = 3$, is the Moore-Read state as shown in [4].

111 Case. The increase of distance between the two layers transforms the 111 state at the bilayer total filling factor $\nu_T = 1$ into two Fermi liquids, each with a filling factor equal to

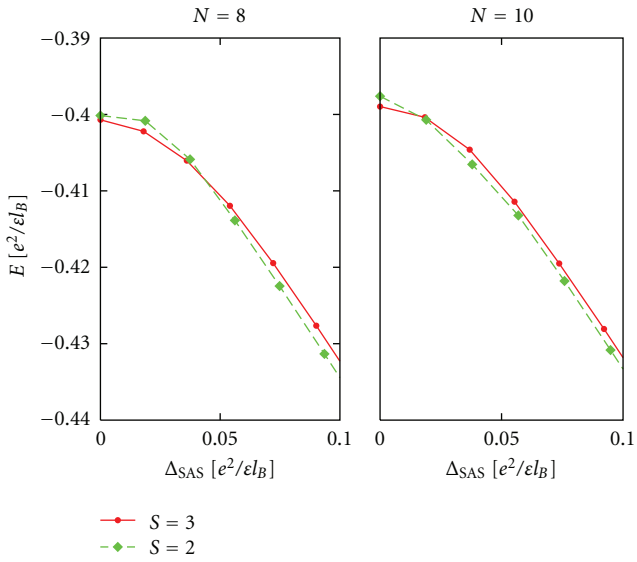

FIGURE 1: (color online) Ground state energies on the sphere as a function of tunneling for the filling factor $\nu_T = 1/2$ and the shifts $\mathcal{S} = 3$ and $\mathcal{S} = 2$. We show data for $N = 8$ and $N = 10$ electrons and the distance is fixed at $d = l_B$.

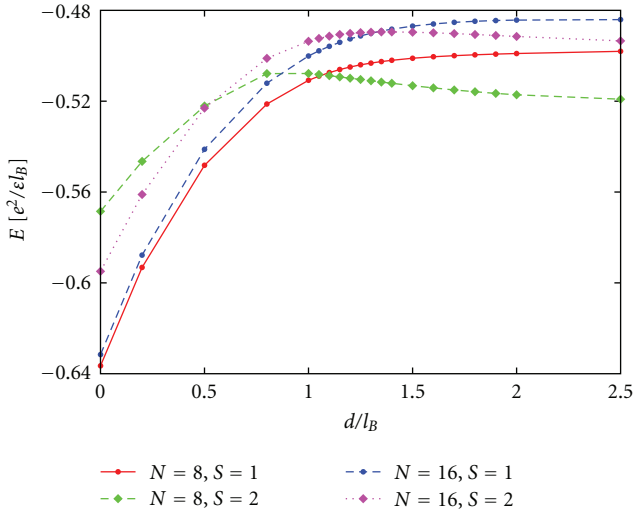

FIGURE 2: (color online) Ground state energies on the sphere as a function of distance for $N = 8$ and $N = 16$ electrons at the filling factor $\nu_T = 1$ and the shifts $\mathcal{S} = 1$ and $\mathcal{S} = 2$.

1/2. An intermediate state based on p-wave pairing between composite fermions of the two layers is a viable candidate, as shown by the numerical investigation in the spherical geometry [10, 11] and analytical work [12]. In the latter investigation the manner of superfluid disordering with the increasing distance was analyzed and implications for the form of the ground state that evolves with distance were derived. Based on the universal phonon correction [18, 19] for small distances, the nature of pairing in the ground state wave function was extrapolated. The most likely kind of pairing was found to be $g(\mathbf{r}) = \sqrt{z/z^*}$, that is, a pure phase that leads to very weak pairing in the opposite angular momentum channel. This result was derived following the long-distance, effective (Chern-Simons) argument with the universal phonon correction. It is different from what we are used to when considering the Pfaffian for which $g(\mathbf{r}) = 1/z$, because here we have the pairing for the opposite value of the angular momentum. The numerics [10, 11] confirms the value of the angular momentum for the pairing. However, as the numerics was done in the LLL, it also does not preclude the pairings of the same angular momentum like $g(\mathbf{r}) = 1/z^*$, which amounts to the same leading short-distance behavior $\tilde{g}(\mathbf{r}) \sim z$, in the LLL, which was determined by Möller et al. [10, 11]. In the following, we will consider only these two possibilities for the pairing, which is expected to be weak (as opposed to the extreme case of strong pairing when the Cooper pairs are exponentially bound).

The physical picture that emerges from these works and the proposal for mixed states of composite bosons and composite fermions in [20] is that, with increasing distance, composite fermions nucleate in the 111 (composite boson) condensate and they do that by pairing in the p-wave pairs. The number of composite fermions or, equivalently, neutral fermions increases with the increase of distance between the layers. Therefore, their chemical potential increases as well. In this case, we can apply the theory of Read and Green, generalized to two species (spinful pairing), which differs from the one we described earlier only by the degeneracy equal to two of the excitations. For $g(\mathbf{r}) = \sqrt{z/z^*}$ pairing, we have $|\Delta_{\mathbf{k}}| \sim |\mathbf{k}|^2$ (due to the very weak pairing), and the theory is always unstable, leading to a direct transition from 111 state into Fermi liquids. For $g(\mathbf{r}) = 1/z^*$, we have $|\Delta_{\mathbf{k}}| \sim |\mathbf{k}|$ and the considerations that we pointed out in the beginning follow. Because the chemical potential of composite fermions increases, we may expect the transition into two Fermi liquids at a finite distance. A comparison of the lowest energies in the spherical geometry for the shifts $\mathcal{S} = 1$ (the 111 state) and $\mathcal{S} = 2$ (the Fermi liquids) corroborates this expectation; see Figure 2. We show data for $N = 8$ and $N = 16$ electrons; the former represents the case where composite fermions in each layer form a filled shell, leading to a local energy minimum [21], while the latter is the largest system attainable by exact diagonalization. Further information on the finite-size scaling of the data, including the estimate for critical d_C, will be given elsewhere [21].

We thus conclude that, even in the clean system with translational invariance such as in Figure 2, we may have a direct transition between the 111 state and the decoupled Fermi liquids occurring at a finite d_C. This happens when the shift $\mathcal{S} = 2$ becomes energetically favorable. If one continues to study the system for $d > d_C$ at the fixed shift of $\mathcal{S} = 1$, a paired state is likely to have an excellent overlap but it will only describe the excited state of the system.

4. Conclusions and Discussion

We described a possible role of the p-wave-paired states in the transition of the 111 and 331 correlated states into Fermi liquids. In the case of the 111 state, we have a transition

into two Fermi liquids induced by increasing the distance between two layers and in the case of the 331 state the transition into a one-component Fermi liquid is induced by tunneling. The previous studies [4, 10, 11] have shown that the p-wave state is slowly nucleated in a correlated state and here we demonstrated that after the transition it may assume the role of an excited state of the Fermi liquid(s). Whether we have a case for a second-order transition in each of our two examples we cannot claim. However, this scenario will likely lead to the observance of smooth transitions into Fermi liquids in experiments as found in [22, 23] in the case of the 111 excitonic, correlated state. As we pointed out in the 331 case by changing the distance between layers, that is, fixing a parameter for the transition driven by tunneling, Pfaffian may indeed become a ground state in the place of the Fermi liquid state [4]. Unfortunately, it becomes so with a very small gap [4, 24] which would again lead to the observance of a smooth transition into a compressible thermodynamic phase in experiments if the beginning phase is the 331 correlated phase.

In the transitions we discussed that are followed by p-wave states, there is a mismatch between the shift numbers characteristic of the beginning of the correlated phase and the ending of the Fermi liquid phase. Quantum fluctuations in the correlated state produce the p-wave state which inherits and accommodates the shift of the correlated phase and therefore has to play a role at the transition into a new phase. Unfortunately, p-wave states are unstable or very weak and give a way to Fermi liquids.

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