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ORIGINAL PAPER

Complexified quantum and classical mechanics

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Abstract: The anti-self-adjoint operators of coordinate and momentum are introduced and applied to the study of tunnelling through the potential barrier, in which the imaginary value of momentum unavoidably appears. Tunnelling through a temporal barrier is treated similarly and it is shown that the quantum system can tunnel, while the classical systems are destroyed by a such barrier. The imaginary observables of coordinate and momentum are used in a novel treatment of passage through the event horizon. By considering the event horizon as the border of the finite potential barrier, it is found that there is no singularity at the centre of a black hole.

Keywords: Anti-self-adjoint operator; Tunnelling; Potential barrier; Event horizon; Virtual particle; Negative energy

1. Introduction

Tunnelling through the potential barrier is a typical quantum phenomenon and there are many articles discussing it from different angles and applying it in a huge variety of diverse situations [1–15]. Recently, attention are attracting the cases of time-dependent potentials, see for instance [16–19]. Usually, the potential depending on the coordinate is multiplied by the time-dependent function and this modulation of a barrier raises interesting effects in different areas of research. Here, instead of proceeding in this direction, we shall analyse the behaviour of the quantum system when it is confronted with a barrier that depends solely on time. Since we are interested in conceptual aspects of the tunnelling through the temporal barrier, we shall discuss it in general, being explicit only in the most simple case of a square barrier. However, we shall compare the behaviour of the quantum system with the behaviour of the classical system in the same situation, and this will be done in order to underline the big difference between the two. So to say, the quantum system can survive, by tunnelling through the barrier, while the classical one cannot survive the action of the temporal barrier.

There are solutions of the Schrödinger equation for which the imaginary values of momentum appear. That is, by applying the self-adjoint operator of momentum on such solutions, one ends up with an imaginary value. This surprising feature was discussed in [20, 21]. The typical examples of this are the solutions of the Schrödinger equation in the case of the square potential, when one considers the solutions in the region where the quantum system is within the barrier, *i.e.*, during its supposed tunnelling through the barrier. To end up with an imaginary value after acting with the operator of the momentum on the solution of the Schrödinger equation is strange since this operator is self-adjoint, so only real values are expected. Directly attached to these imaginary values of momentum are negative values of kinetic energy, which are equally well unacceptable from the point of view of our everyday experience. The same situation will arise also in the case of the time-dependent barrier that we are going to consider here. However, we shall approach these imaginary values in a completely non-standard way. Namely, they will lead us, so to say, to the complexification of the coordinate and momentum. Just like beside the real axis one introduces the imaginary one, we shall introduce anti-self-adjoint (skew-symmetric) operators of coordinate and momentum. These new operators will have imaginary eigenvalues and offer an adequate and consistent description of the tunnelling through barriers.

The formalism of the operator of time, that we have proposed in [22–26], will be used here. Let us just mention that there is a whole variety of topics and approaches related to the operator of time, e.g., [27–29] and references therein. However, our approach is similar to [30], and references therein, and [31], and its crucial point is to treat time and energy on an equal footing with coordinate and

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momentum. This means that the separate Hilbert space, in which operators of time and energy act, is introduced, just as it is done for each degree of freedom in the standard formulation of quantum mechanics. By doing this, the Pauli's objection is avoided. Proceeding in this way, the same commutation relation that holds for the coordinate and momentum is imposed for the energy and time, which leads to an unbounded spectrum of these operators. Finally, the Schrödinger equation appears as a constraint in the overall Hilbert space, selecting the states with non-negative energy for the standard Hamiltonians. However, after the algebra of anti-self-adjoint operators of coordinate and momentum is introduced, the negative eigenvalues of energy will naturally appear in the formalism.

The standard quantum mechanics, with its self-adjoint operators of coordinate and momentum and appropriate real eigenvalues, is a part of the complete theory that involves anti-self-adjoint operators of coordinate and momentum and their imaginary eigenvalues. The last ones, together with the negative energies, find natural explanation within the complexified quantum mechanics. So, we find not only that there is nothing wrong with the imaginary values of momentum in the case of tunnelling, but that these imaginary values are meaningful since, besides our part of the universe characterized by real numbers, there is also the other one characterized by imaginary numbers. These two parts are on an equal footing and together form the complete universe.

As an example of the possible application of complexified mechanics, we shall propose a new way of looking at passages of the quantum and classical systems through the event horizon during the fall in and possible escape from a black hole. We will discuss the evaporation of a black hole and the virtual particles that escaped from a black hole.

2. Operators of time and energy

Analogously to the treatment of spatial degrees of freedom, a separate Hilbert space $\mathcal{H}_t$, in which operators of time $\hat{t}$ and energy $\hat{s}$ act non-trivially, can be introduced. So, for the case of one degree of freedom, there are self-adjoint operators $\hat{q}\otimes\hat{I}$, $\hat{p}\otimes\hat{I}$, $\hat{I}\otimes\hat{t}$ and $\hat{I}\otimes\hat{s}$, acting in $\mathcal{H}_q\otimes\mathcal{H}_t$, with commutation relations:

$$\frac{1}{i\hbar}[\hat{q}\otimes\hat{I},\hat{p}\otimes\hat{I}]=\hat{I}\otimes\hat{I}, \tag{1}$$

$$\frac{1}{i\hbar}[\hat{I}\otimes\hat{t},\hat{I}\otimes\hat{s}]=-\hat{I}\otimes\hat{I}. \tag{2}$$

The other commutators vanish. The operators of time $\hat{t}$ and energy $\hat{s}$ have continuous spectrum $\{-\infty,+\infty\}$, just like the operators of coordinate and momentum $\hat{q}$ and $\hat{p}$. The eigenvectors of $\hat{t}$ are $|t\rangle$ for every $t\in\mathbf{R}$. In $|t\rangle$ representation, operator of energy is given by $i\hbar\frac{\partial}{\partial t}$ and its eigenvectors $|E\rangle$, in the same representation, are $e^{\frac{1}{i\hbar}E\cdot t}$ for every $E\in\mathbf{R}$. In [22] we have shown how the unbounded spectrum of the operator of energy is regulated by the Schrödinger equation. Namely, the Schrödinger equation, which appears as a constraint in $\mathcal{H}_q\otimes\mathcal{H}_t$, selects the states with non-negative energy since the spectrum of the standard Hamiltonians is bounded from below. That is, the Hamiltonian and operator of energy are acting in different Hilbert spaces, but there is a subspace of the total Hilbert space in which:

$$\hat{s}|\psi\rangle=H(\hat{q},\hat{p})|\psi\rangle. \tag{3}$$

The states that satisfy this equation have non-negative energy for the usually used Hamiltonians. The last equation is nothing else but the Schrödinger equation. By taking its $|q\rangle\otimes|t\rangle$ representation, one gets the familiar form of the Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}\psi(q,t)=\hat{H}\psi(q,t), \tag{4}$$

where we introduced the shorthand notation $\hat{H}=H(q,-i\hbar\frac{\partial}{\partial q})$. In other words, the operator of energy has negative eigenvalues as well as non-negative, but the states with non-negative energies, used in the standard quantum mechanics, are selected by the Schrödinger equation due to the non-negative spectrum of $H(\hat{q},\hat{p})$. For the time-independent Hamiltonian, the typical solution of the Schrödinger equation $\psi_E(q)e^{\frac{1}{i\hbar}E\cdot t}$ is $|q\rangle\otimes|t\rangle$ representation of $|\psi_E\rangle\otimes|E\rangle$, where $H(\hat{q},\hat{p})|\psi_E\rangle=E|\psi_E\rangle$ and $\hat{s}|E\rangle=E|E\rangle$. The energy eigenvectors $|E\rangle$ have the same formal characteristics as, say, the momentum eigenvectors (they are "normalized" to $\delta(0)$ and, for different values of energy, they are mutually orthogonal).

The negative eigenvalues of $\hat{s}$ will appear in the discussion below.

3. Spatial barrier

Let us briefly review the standard treatment of the tunnelling through the square potential. If the potential vanishes everywhere except between q_a and q_b, where has constant value $V=V_0$, one firstly solves the Schrödinger equation for the regions where $V=0$ and then for the region where $V=V_0$. If the energy of the quantum system is taken to be E_0, then the solutions outside the barrier, in the coordinate representation, are plane waves $e^{\pm\frac{1}{i\hbar}p\cdot q}e^{\frac{1}{i\hbar}E_0\cdot t}$, where $E_0=\frac{p^2}{2m}$. For the region with $V=V_0$ one finds $e^{\pm\frac{1}{i\hbar}p'\cdot q}e^{\frac{1}{i\hbar}E\cdot t}$, where $E_0=\frac{p'^2}{2m}+V_0$. We are talking about tunnelling when $E_0 < V_0$. Then, the quantum system can be in the region that is forbidden from the point of view of classical mechanics. Namely, the classical system, having momentum p, will approach the potential barrier, say from the left, and at the point $q = q_a$ it will be pushed back by the force caused by the barrier. It will never enter the region between q_a and q_b since it does not have enough energy to overcome the repulsive force that acts at q_a.

After finding mentioned solutions for all three regions, one proceeds with making superpositions of the appropriate solutions (which are occasionally called left and right moving waves). The coefficients in these superpositions are found from the boundary conditions of the wave function at the barrier surface. Namely, the requirement that the wave function be continuous and smooth at the boundaries q_a and q_b, the meaning of which is that one demands the wave function and its derivative to be continuous, leads one to the solution of the considered Schrödinger equation for the whole space. Discussion regarding the tunnelling through the potential barrier usually proceeds by the calculation of reflection and transition rates. Instead of repeating this in detail, let us focus on the difference between the classical and quantum systems that is essential for tunnelling. Namely, regarding the position, the classical system is always located at one point. Its state, *i.e.*, the coordinate part of its state, is given by $\delta(q - q(t))$. So, at some moment, the classical system starts to feel the presence of the barrier. More precisely, only at the moment when it comes to q_a it feels the repulsive force that pushes it back to the region from which it came. In difference to the classical system, the quantum system, having energy E_0, is not located in one point of the space. Its state spreads all over the space, and in the standard approach, as outlined above, it is given by mentioned superpositions of $e^{\pm\frac{1}{i\hbar}p\cdot q}$, for q between $-\infty$ and q_a, $e^{\pm\frac{1}{i\hbar}p'\cdot q}$, for q between q_a and q_b, and $e^{\pm\frac{1}{i\hbar}p\cdot q}$, for q between q_b and $+\infty$. In the present context, being in a state with a sharp value of energy actually means that the quantum system is not in a state that is spatially located. Since we are not measuring the coordinate of the quantum system, we are not reducing its state to the one with the sharp value of the position, in particular not to the state $|q_a\rangle$. In this way, the quantum system is never actually in front of the barrier, so it does not feel its repulsive force. So to say, being in a state with a sharp value of energy, the quantum system is potentially in front, inside and beside the barrier simultaneously, and its existence is not concentrated to some particular point of the space.

4. Imaginary momentum

If one applies momentum operator $-i\hbar\frac{\partial}{\partial q}$ to the states of quantum system $e^{\pm\frac{1}{i\hbar}p'\cdot q}e^{\frac{1}{i\hbar}E\cdot t}$, where $E_0 = \frac{p'^2}{2m} + V_0$, then for $E_0 < V_0$ one ends up with the imaginary value of momentum. This is, of course, strange since we do not expect to find imaginary values when the self-adjoint operators act on perfectly correct solutions of the Schrödinger equation. There are other situations for which we find imaginary momentum, as well, see [32].

Since the imaginary values are unavoidable in the considered cases, then their meaning should be addressed. As is well known, Hermitian and self-adjoint operators are used in quantum mechanics because they have real eigenvalues. The world we live in is characterized by real numbers as the values of considered quantities. So, if we say, in a case when some system is in a state that has real value of some observable, that this quantity belongs to our real world, then in the case when the system is in a state that gives imaginary value of some observable we cannot attribute reality to this quantity. This quantity with imaginary value does not belong to the real world. It belongs to the world of imaginary quantities.

The unavoidable imaginary values of the momentum for $e^{\pm\frac{1}{i\hbar}p'\cdot q}e^{\frac{1}{i\hbar}E\cdot t}$ (where $E_0 = \frac{p'^2}{2m} + V_0$ and $E_0 < V_0$) are demonstrations of the existence of a world other than ours. So, there are two worlds. One is the world characterised by properties that are real. It is our world where everything is expressed by real numbers, but there is another world where everything is expressed by imaginary numbers. This world of unreality is on an equal footing with ours. It is as natural as our world. For the sake of simplicity, let us call that other world the imaginary world. The universe consists of both the real and the imaginary world.

5. Anti-self-adjoint operators

As we have underlined, it is unavoidable to end up with the imaginary values of the momentum for the above-mentioned states. Since this is not consistent with the fact that momentum is a self-adjoint operator, the question is whether there exists an operator of the momentum with which we can reproduce all important features of the tunnelling considered, being such that the imaginary values are in accordance with its nature. The answer is affirmative. Namely, there are anti-self-adjoint operators which have imaginary eigenvalues. For such an operator it is completely consistent to obtain the imaginary value when applying this operator on some state, just as it is consistent to get the real value when a self-adjoint operator is applied to some state.

Let us introduce anti-self-adjoint operators of the imaginary coordinate and the imaginary momentum in the following way. For each real q_{re} there is $|q_{re}\rangle$, which is the eigenvector of the self-adjoint operator of real coordinate $\hat{q}_{re} = \int q_{re}|q_{re}\rangle\langle q_{re}|dq_{re}$:

$$\hat{q}_{re}|q_{re}\rangle = q_{re}|q_{re}\rangle. \tag{5}$$

The self-adjoint operator of the real momentum $\hat{p}_{re}$, in the basis $|q_{re}\rangle$, is represented by $-i\hbar\frac{\partial}{\partial q_{re}}$. Its eigenvectors, in the same representation, are the (real) plane waves $e^{\frac{1}{i\hbar}p_{re}\cdot q_{re}}$ with the real eigenvalues. These two operators, acting in the rigged Hilbert space $\mathcal{H}_{re}$, are the standard operators of coordinate and momentum used in quantum mechanics. They form the Heisenberg algebra and do not commute $\frac{1}{i\hbar}[\hat{q}_{re}, \hat{p}_{re}] = \hat{I}_{re}$, where $\hat{I}_{re} = \int |q_{re}\rangle\langle q_{re}|dq_{re}$.

Besides this, there is the algebra of non-commuting anti-self-adjoint operators of the coordinate $\hat{q}_{im}$ and momentum $\hat{p}_{im}$, acting in the rigged Hilbert space $\mathcal{H}_{im}$. The spectral form of the imaginary coordinate $\hat{q}_{im}$, in the basis $|q_{im}\rangle$, where q_{im} ranges over entire imaginary axis, is $\hat{q}_{im} = \int q_{im}|q_{im}\rangle\langle q_{im}|dq_{im}$ (with the real measure dq_{im}), and its action on the eigenvectors is given by:

$$\hat{q}_{im}|q_{im}\rangle = q_{im}|q_{im}\rangle. \tag{6}$$

In the basis $|q_{im}\rangle$, the anti-self-adjoint operator of the imaginary momentum is represented by $-i\hbar\frac{\partial}{\partial q_{im}}$. Its eigenvectors, in the same representation, are the imaginary plane waves $e^{\frac{1}{i\hbar}p_{im}\cdot q_{im}}$ with the imaginary eigenvalues. Analogously to the case of the real coordinate and momentum, the commutator of the imaginary ones is proportional to the $\hat{I}_{im}$ which is $\hat{I}_{im} = \int |q_{im}\rangle\langle q_{im}|dq_{im}$.

In the case of one degree of freedom, the complete description demands the direct product of the rigged Hilbert spaces $\mathcal{H}_{re}$ and $\mathcal{H}_{im}$. In $\mathcal{H}_{re} \otimes \mathcal{H}_{im}$ the operator of the real coordinate is $\hat{q}_{re} \otimes \hat{I}_{im}$, while the operator of the imaginary coordinate is $\hat{I}_{re} \otimes \hat{q}_{im}$ (and similarly for the operators of momentum). These operators, together with the identity, form the basis of complexified quantum mechanics.

The quantum and classical mechanics have to be identical regarding the formal structures. Therefore, there are the imaginary variables of coordinate and momentum q_{im} and p_{im}, beside the standard real ones q_{re} and p_{re}. These variables will be used below.

6. Temporal barrier

The opportunity to be in a superposition of states is essential for tunnelling through the spatial barrier. When the quantum system is in a state with a sharp value of momentum it is potentially everywhere without actually being there. This is related to the non-commutativity of the operators of coordinate and momentum. On the other side, the classical system simultaneously possesses sharp values of coordinate and momentum since they commute. A similar situation is with energy and time. The operators $\hat{s}$ and $\hat{t}$ do not commute, so they do not have common eigenvectors. Sharp values of energy imply no definite tunnelling time. The classical system, of course, is always present and has definite energy since the variables s and t commute. As in the case of the spatial barrier, this difference between classical and quantum systems is what makes possible tunnelling through the temporal barrier.

How the temporal barrier would affect the classical system can be understood by considering the following situation. Let the initially free classical system interacts with some other one in such a way that it loses its mechanical energy over time, transferring it to that other system. Within the formalism of complexified classical mechanics, let the total energy of these systems be:

$$\frac{p_{re}^2}{2m} + \frac{p_{im}^2}{2m} + W(t), \tag{7}$$

where $W(t)$ is the energy of the other system for which we assume that, at t_1, it starts to rise smoothly from zero, overcomes the initial energy of the considered system, and then, after some time, smoothly drops down becoming zero at t_2. As it rises, the energy of the classical system lowers in order to keep the total energy of these systems constant. At some moment t_a, $t_1 < t_a$, $W(t)$ becomes equal to the energy that the system had before t_a, so the energy of the system becomes equal to zero. At some t_b, $t_a < t_b$, $W(t)$ drops to the value of the initial energy of the system, and then at t_2, $t_b < t_2$, vanishes. If the classical system initially was in the state with $p_{re} = p_{re}^0$ and $p_{im} = 0$, so all of its energy is in the positive kinetic energy, it is crucial to find out what happens at t_a.

In between t_1 and t_a system loses positive kinetic energy of the motion in the real world, at t_a it becomes zero and then negative since $W(t)$ rises. What this means can be better understood in the full relativistic approach. So, the energy of the system is given by:

$$\frac{mc_{re}^2}{(1 - \frac{v_{re}^2}{c_{re}^2})^{\frac{1}{2}}} + \frac{mc_{im}^2}{(1 - \frac{v_{im}^2}{c_{im}^2})^{\frac{1}{2}}}, \tag{8}$$

where $c_{re} = c$ is the speed of light in the real world and c_{im} is the speed of light in the imaginary world. (The speed of light in the imaginary world is imaginary, as are all velocities, and $c_{im} = i \cdot c$.) In between t_1 and t_a, due to the fourth component of the 4-vector of force in the real world, which is equal to $-\frac{1}{c_{re}}\frac{dW(t)}{dt}$, the fourth component of the 4-vector of momentum in the real world decreases until it becomes mc_{re} at t_a. Since $W(t)$ continues to increase, the further decrease of the system's energy cannot proceed by the decrease of its rest mass m. Namely, the particle at rest in both worlds has total energy:

$$mc_{re}^2 + mc_{im}^2 = 0. \tag{9}$$

This holds irrespective of the value of m. So, from the state with the vanishing energy at t_a, the system goes to the state with a small negative kinetic energy of the motion in the real world:

$$-\frac{mc_{re}^2}{(1-\frac{v_{re}^2}{c_{re}^2})^{\frac{1}{2}})} - mc_{im}^2. \tag{10}$$

This means that the mass of the system becomes negative. With the negative rest mass, the fourth component of the 4-vector of momentum in the real world is inverted. And, if the positive fourth component of the 4-vector of momentum in the real world, which was the case before t_a, caused the propagation in the positive direction of the fourth component of the 4-vector of position in the real world $c_{re}t$, then the negative one causes propagation in the negative direction. As is well known, having negative mass is equivalent to the propagation backward in time, so at t_a, by acquiring the negative mass, the system starts to propagate towards the past. This holds for both the real and the imaginary world.

On the other hand, the quantum system, like in the case of a spatial barrier, can tunnel this temporal barrier. If before t_1 it was in the state $|p_{re}^0\rangle$ and $|p_{im}^0\rangle$, where $p_{im}^0 = 0$, then it can be observed after t_2. As stated before, at the moment that is the turning point for the classical system, the quantum one would not be present since, for all the time until it is observed, it would be in an eigenstate of energy. So to say, being in a state with a sharp value of energy, it potentially exists before, during and after the period in which the barrier is present, and the existence of the quantum system is not connected to any moment of time in particular - uncountably many moments of time are superposed in the actual state of the system with a sharp value of the energy. Its $|p_{re}(t)\rangle$ and $|p_{im}(t)\rangle$ would vary with respect to t in such a way that the energy of the system plus $W(t)$ is equal to the initial $\frac{p_{re}^{0^2}}{2m}$. Concretely, in the case when $W(t)$ is square shaped between t_a and t_b, with the value W_0, $W_0 > \frac{p_{re}^{0^2}}{2m}$, the state of the quantum system before the barrier is:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{t_a} e^{\frac{-1}{i\hbar}p_{re}^0\cdot q_{re}} e^{\frac{-1}{i\hbar}p_{im}^0\cdot q_{im}} e^{\frac{1}{i\hbar}E_0\cdot t}|q_{re}\rangle \otimes |q_{im}\rangle \otimes |t\rangle dq_{re}dq_{im}dt, \tag{11}$$

where $p_{im}^0 = 0$ and $E_0 = \frac{p_{re}^{0^2}}{2m}$. Between the moments t_a and t_b the state is:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{t_a}^{t_b} e^{\frac{-1}{i\hbar}p_{re}^T\cdot q_{re}} e^{\frac{-1}{i\hbar}p_{im}^T\cdot q_{im}} e^{\frac{1}{i\hbar}E_T\cdot t}|q_{re}\rangle \otimes |q_{im}\rangle \otimes |t\rangle dq_{re}dq_{im}dt, \tag{12}$$

where p_{re}^T, p_{im}^T and E_T are the tunneling momenta and energy, and where $p_{re}^T = 0$, $p_{im}^T = (2\,m(\frac{p_{re}^{0^2}}{2\,m} - W_o))^{\frac{1}{2}}$ and $E_T = \frac{p_{im}^{T^2}}{2\,m}$. Finally, after t_b the state is:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{t_b}^{+\infty} e^{\frac{-1}{i\hbar}p_{re}^0\cdot q_{re}} e^{\frac{-1}{i\hbar}p_{im}^0\cdot q_{im}} e^{\frac{1}{i\hbar}E_0\cdot t}|q_{re}\rangle \otimes |q_{im}\rangle \otimes |t\rangle dq_{re}dq_{im}dt. \tag{13}$$

In both cases of the spatial and the temporal barrier, the appropriate momentum of the classical system is reverted at the point where the system is confronted with the barrier. However, there is a big difference between these two cases. Since there is no back propagation in time, at the moment t_a the classical system ceases to exist, it is no longer present at later times. It can be said that the force executed by the barrier is the cause of its death and destruction. (Dead means to be at rest.) In this context, for the quantum system, it could be said that tunnelling is the process of dying at t_a in the real world and immediate reincarnation in the imaginary world and, at the moment t_b, resurrection in the real world.

7. Black hole modelled as a barrier

When a black hole is looked at from its centre it looks like a barrier. The attractive gravitational force, directioned towards the centre, plays the same role as the repulsive force of the potential barrier considered above. For the potential barrier, it is important to know at which point the barrier starts and how high it can be.

The particle in the gravitational field at rest at $r = +\infty$ has vanishing energy [33]. During its fall into a black hole, its velocity increases. At the distance r^E from the centre, where r^E is the Schwarzschild radius, the velocity of the particle becomes equal to the speed of light. Since the speed of light sets the upper bound for all velocities, the particle cannot move faster. This means that at the r^E is the bottom of the potential well or, when looked at from the centre of a black hole, at r^E starts the potential barrier. The depth of the potential well is finite and equal to the value of the gravitational potential at r^E, which means that the potential is constant from $r = 0$ to r^E. In other words, the gravitational force equals zero within the event horizon - the sphere of radius r^E, and the speed of light, as the finite upper bound for velocities, determines the depth of the potential well.

After its passage through the event horizon, the particle thermalises with the matter that has been already present in a black hole, so its velocity drops in time. Due to the absence of the gravitational force within the sphere of radius r^E, the matter is not shrinking to a single point and, therefore, there is no singularity at $r = 0$ (all the matter contained within the event horizon is not concentrated within the single point).

If a classical system, with velocity v, $v < c$, that is directioned outwards a black hole, tries to escape it, it would be pulled back by attractive gravitational force. In the case of the potential barrier considered above, the repulsive force and insufficient energy were the reasons why the system could not go beyond a certain point, while here the attractive force, directioned towards the centre of a black hole, and insufficient energy are the reasons why the system cannot escape. The quantum system also behaves as it behaved in the case of the potential barrier.

In the case of the square spatial barrier the state of the quantum system is:

$$\begin{aligned}
&\int_{-\infty}^{q_a}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{\frac{-1}{i\hbar}p_{re}^0\cdot q_{re}} e^{\frac{-1}{i\hbar}p_{im}^0\cdot q_{im}} e^{\frac{1}{i\hbar}E_0\cdot t}|q_{re}\rangle \\
&\otimes |q_{im}\rangle \otimes |t\rangle dq_{re}dq_{im}dt+ \\
&+\int_{q_a}^{q_b}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{\frac{-1}{i\hbar}p_{re}^T\cdot q_{re}} e^{-\frac{1}{i\hbar}p_{im}^T\cdot q_{im}}\cdot e^{\frac{1}{i\hbar}E_0\cdot t}|q_{re}\rangle \\
&\otimes |q_{im}\rangle \otimes |t\rangle dq_{re}dq_{im}dt+ \\
&+\int_{q_b}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{\frac{-1}{i\hbar}p_{re}^0\cdot q_{re}} e^{\frac{-1}{i\hbar}p_{im}^0\cdot q_{re}} e^{\frac{1}{i\hbar}E_0\cdot t}|q_{re}\rangle \\
&\otimes |q_{im}\rangle \otimes |t\rangle dq_{re}dq_{im}dt,
\end{aligned} \tag{14}$$

where p_{re}^0 is the initial real momentum, $p_{im}^0 = 0$, $p_{re}^T = 0$ and $p_{im}^T = (2\,m(\frac{p_{re}^{0^2}}{2\,m} - V_o))^{\frac{1}{2}}$. The quantum system can escape from a black hole, which is modelled here as a barrier, by tunnelling through it. Its state in this situation is:

$$\begin{aligned}
&\int_{0}^{r_a}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{\frac{-1}{i\hbar}p_{re}^0\cdot r_{re}} e^{\frac{-1}{i\hbar}p_{im}^0\cdot q_{im}} e^{\frac{1}{i\hbar}E_0\cdot t}|r_{re}\rangle \\
&\otimes |q_{im}\rangle \otimes |t\rangle dr_{re}dq_{im}dt+ \\
&+\int_{r_a}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{\frac{-1}{i\hbar}p_{re}^T\cdot r_{re}} e^{-\frac{1}{i\hbar}p_{im}^T\cdot q_{im}}\cdot e^{\frac{1}{i\hbar}E_0\cdot t}|r_{re}\rangle \\
&\otimes |q_{im}\rangle \otimes |t\rangle dr_{re}dq_{im}dt,
\end{aligned} \tag{15}$$

where p_{re}^0 is the initial real $p_{im}^0 = 0$, $p_{re}^T = 0$ and $p_{im}^T = (2\,m(V(r^E) + \frac{p_{re}^{0^2}}{2\,m} - V(r_{re}))^{\frac{1}{2}}$ and $V(r_{re}) = -\gamma\frac{m\cdot M}{r_{re}}$. So, due to the possibility to tunnel the barrier, the quantum system could be found outside the event horizon. For the systems that have evaporated from a black hole it is usually said that they are virtual particles. The attribute virtual is used in order to underline that their kinetic energies need not be positive. Within the complexified quantum mechanics it becomes obvious that the evaporated particles are the standard particles that have the non-vanishing imaginary momentum, propagating forwards, not backwards, in time (as is the case for some Feynman diagrams).

8. Conclusions

The complexified quantum and classical mechanics were founded by the introduction of the anti-self-adjoint operators of coordinate and momentum for the quantum system and the imaginary variables of the classical system. Within the proposed formalism, the tunnellings through the barriers, spatial and temporal, were thoroughly discussed. It is a well-known fact that, during the tunnelling through the potential barrier, the quantum system is characterized by the imaginary value of the momentum. This appears to be strange within the framework of standard quantum mechanics since only the self-adjoint operator of momentum is used, whose spectrum is real. The complexified quantum mechanics offered a self-consistent description of the tunnelling through the spatial barrier where the unavoidable imaginary value of the momentum does not contradict the nature of the operator of the imaginary momentum. The enlarged quantum mechanics was used in the discussion of the tunnelling through the temporal barrier and, in order to make the argumentation more transparent, the previously introduced operator of time formalism was employed. The complexified version of classical mechanics, on the other hand, was used in the discussion regarding the system's passage through the event horizon, which was found to be the border of the potential barrier around a black hole.

The main conclusions are the following. The tunnelling through the potential barrier shows that the standard formulation of quantum mechanics, and consequently of classical mechanics as well, have to be enlarged. The quantum system can tunnel through the temporal barrier, while such a barrier destroys the classical systems. There is no singularity at the centre of a black hole and the negative energies are natural for the imaginary world.

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