Quench Dynamics of Harmonically Trapped Ideal Bosons

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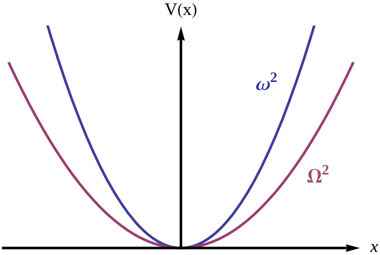

Outline

- One Particle
- *N* Particles
- Partition Function
- Reduced Density Matrix
- Outlook

One Particle

Setup

- Time evolution of density operator

$$\hat{\varrho}(t) = e^{-\frac{i}{\hbar}\hat{H}_\Omega t}\ \hat{\varrho}\,(0)\ \ e^{+\frac{i}{\hbar}\hat{H}_\Omega t}$$

- Coordinate representation

$$\varrho_1\,(x_b, x_{b'}; t) = \int dx_a \int dx_{a'}\,(x_b, t|x_a, 0)\ \varrho_1\,(x_a, x_{a'}; \beta)\ (x_{a'}, 0|x_{b'}, t)$$

$$\varrho_1\,(x_a, x_{a'}; \beta) = \frac{1}{Z_1(\beta)}\,(x_a, \beta|x_{a'}, 0)$$

- Imaginary-time evolution amplitude

$$(x_a, \beta|x_{a'}, 0) = \sqrt{\frac{M\omega}{2\pi\,\hbar\sinh\left(\hbar\beta\omega\right)}}\exp\left\{-\frac{M\omega}{2\,\hbar}\left[\frac{\left(x_a^2 + x_{a'}^2\right)\cosh\left(\hbar\beta\omega\right) - 2\,x_a x_{a'}}{\sinh\left(\hbar\beta\omega\right)}\right]\right\}$$

- Real-time evolution amplitude: $(x_a, \beta|x_{a'}, 0) \overset{\hbar\beta\to it}{\longrightarrow} (x_b, t|x_a, 0)$

$$\sinh\left(\hbar\beta\omega\right) \longrightarrow i\sin\left(\omega t\right)$$

$$\cosh\left(\hbar\beta\omega\right) \longrightarrow \cos\left(\omega t\right)$$

One Particle

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$$\sinh\,(\hbar\beta\omega) \longrightarrow i\sin\,(\omega t)$$

$$\cosh\,(\hbar\beta\omega) \longrightarrow \cos\,(\omega t)$$

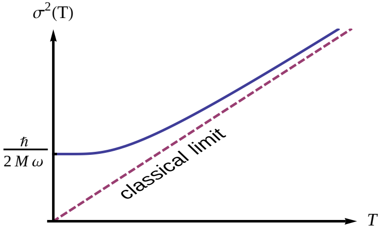
One Particle

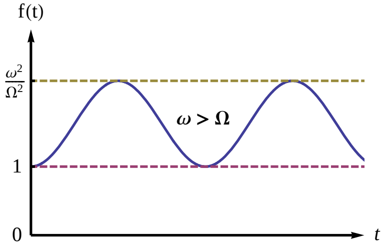
Result and Interpretation

- Perform *2-D* Gaussian integral
- Diagonal elements: $\varrho_1(x_b, x_b; t) = \frac{1}{\sqrt{2\pi\,\sigma_1^2(t;T)}} \exp\left\{-\frac{x_b^2}{2\sigma_1^2(t;T)}\right\}$
- Width $\sigma_1^2(t;T) = \sigma^2(T)\, f(t)$

$$\sigma^2(T) = \frac{\hbar}{2M\omega} \coth\left(\frac{\hbar\beta\omega}{2}\right)$$

$$f(t) = \frac{1}{2}\left(1 + \frac{\omega^2}{\Omega^2}\right) + \frac{1}{2}\left(1 - \frac{\omega^2}{\Omega^2}\right)\cos(2\Omega t)$$





N Particles

Setup and Result

- *N*-particle harmonic oscillator density matrix

$$\varrho_N\left(x_{1a},\ldots,x_{Na};x_{1a'},\ldots,x_{Na'};\beta\right) = \frac{1}{Z_N(\beta)}\left(x_{1a},\ldots,x_{Na};\beta|x_{1a'},\ldots,x_{Na'};0\right)^{\mathrm{s}}$$

- Density matrix

$$\varrho_N(x_{1b},\ldots,x_{Nb};x_{1b'},\ldots,x_{Nb'};t) = \int d^N x_a \int d^N x_{a'}\left(x_{1b},\ldots,x_{Nb};t|x_{1a},\ldots,x_{Na};0\right)^{\mathrm{s}} \times \varrho_N\left(x_{1a},\ldots,x_{Na};x_{1a'},\ldots,x_{Na'};\beta\right)\left(x_{1a'},\ldots,x_{Na'};0|x_{1b'},\ldots,x_{Nb'};t\right)^{\mathrm{s}}$$

- *N*-particle amplitudes

$$\left(x_{1b},\ldots,x_{Nb};t|x_{1a},\ldots,x_{Na};0\right)^{\mathrm{s}} = \frac{1}{N!}\sum_P\left(x_{P(1)b},t|x_{1a},0\right)\cdots\left(x_{P(N)b},t|x_{Na},0\right)$$

- Evaluation yields with one-particle density matrix:

$$\varrho_N(x_{1b},\ldots,x_{Nb};x_{1b'},\ldots,x_{Nb'};t) = \frac{Z_1^N(\beta)}{Z_N(\beta)}\frac{1}{N!}\sum_P \varrho_1(x_{P(1)b},x_{1b'};t)\cdots\varrho_1(x_{P(N)b},x_{Nb'};t)$$

Partition Function

Setup

- Trace of density matrix: Normalisation
  $$1 = \int d^N x_b \ \varrho_N(x_{1b}, \ldots, x_{Nb}; x_{1b}, \ldots, x_{Nb}; t)$$
- Abbreviate: $a = \dfrac{M\omega}{\hbar f(t)} \coth(\hbar\beta\omega) \qquad b = \dfrac{M\omega}{\hbar f(t)} \sinh(\hbar\beta\omega)^{-1}$

$$Z_N(\beta) \sim \frac{1}{N!} \sum_P \int d^N x_b \ \exp\left\{-a\left(x_{1b}^2 + \cdots + x_{Nb}^2\right) + b\left(x_{1b}\, x_{P(1)b} + \cdots + x_{Nb}\, x_{P(N)b}\right)\right\}$$

- N-Dimensional Gaussian integral: $\int d^N x \ \exp\left\{-\mathbf{x}^\dagger \mathbf{A}(P)\mathbf{x}\right\} = \sqrt{\dfrac{\pi^N}{\det \mathbf{A}(P)}}$

Problem: Permutation dependent matrices: $\mathbf{A}(P) = a\,\delta_{ij} - \frac{b}{2}\left[P_{ij} + P_{ji}\right]$
Determinant?

Solution: $\det \mathbf{A}(P)$ decomposes into product of fixed per-cycle sub-determinants

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Partition Function

Matrices and Determinants

- Use simplest one-cycle matrix to calculate per *n*-cycle subdeterminant

$$\det A(n) = \begin{vmatrix} a & -\frac{b}{2} & 0 & \dots & \dots & -\frac{b}{2} \\ -\frac{b}{2} & a & -\frac{b}{2} & 0 & \dots & 0 \\ 0 & -\frac{b}{2} & a & -\frac{b}{2} & & \vdots \\ \vdots & 0 & -\frac{b}{2} & a & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & -\frac{b}{2} \\ -\frac{b}{2} & 0 & \dots & & -\frac{b}{2} & a \end{vmatrix}$$

- Laplace expansion along first row:

$$\det A(n) = aT_{n-1} + \frac{b}{2}C_{n-1} + (-1)^n \frac{b}{2}D_{n-1}$$

- We only need to know T_n:

$$\det A(n) = T_n - \frac{b^2}{4}T_{n-2} - 2\,(b/2)^n$$

$$T_n = \begin{vmatrix} a & -\frac{b}{2} & 0 & \dots & \dots & 0 \\ -\frac{b}{2} & a & -\frac{b}{2} & 0 & \dots & 0 \\ 0 & -\frac{b}{2} & a & -\frac{b}{2} & & \vdots \\ \vdots & 0 & -\frac{b}{2} & a & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & -\frac{b}{2} \\ 0 & 0 & \dots & & -\frac{b}{2} & a \end{vmatrix}$$

- Recursion relation:

$$T_{n+2} = aT_{n+1} - \left(\frac{b}{2}\right)^2 T_n$$

$$T_1 = a\ , \qquad T_2 = a^2 - \left(\frac{b}{2}\right)^2$$

Partition Function

Recursion Relation and Result

- *Z*-Transform: $\mathcal{Z}\{T_n\} = T(z) = \sum_{n=0}^{\infty} T_n\, z^{-n}\,, \qquad T_n$ sequence
- Our case: $T_n = \frac{1}{2^{n+1}} \left[\frac{(a+\sqrt{a^2-b^2})^{n+1} - (a-\sqrt{a^2-b^2})^{n+1}}{\sqrt{a^2-b^2}} \right]$
- Result: $T_n = \left[\frac{M\,\omega}{\hbar\, f(t)} \right]^n \frac{1}{2^n \sinh^n(\hbar\beta\omega)} \left[\frac{1}{Z_1(n\beta)} \right]^2$
- Inserting yields canonical partition function in cycle representation

$$Z_N(\beta) = \sum_{(C_1,\ldots,C_N)}^{\sum_n C_n n = N} \prod_{n=1}^{\infty} \underbrace{\frac{1}{n^{C_n}\, C_n!}}_{\text{multiplicity factor}} \left[Z_1(n\beta)\right]^{C_n}$$

- n : cycle length, C_n: number of n-cycles

Partition Function

Recursion Relation and Result

- Z-Transform: $\mathcal{Z}\{T_n\} = T(z) = \sum_{n=0}^{\infty} T_n \, z^{-n}$, T_n sequence
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One-Particle Reduced Density Matrix

Setup and Result

- Partial trace ($x_{nb} = x_{nb'}$ for $n = 2, \ldots N$, leave $x_{1b} \neq x_{1b'}$):

$$\varrho_1^{(\mathrm{r})}(x_{1b}, x_{1b'}; t) = \int_{-\infty}^{\infty} d^{N-1}x_b \; \varrho_N(x_{1b}, x_{2b}, \ldots, x_{Nb}; x_{1b'}, x_{2b}, \ldots, x_{Nb}; t)$$

- Pull out broken cycle:

$$\varrho_1^{(\mathrm{r})}(x_{1b}, x_{1b'}; t) = \frac{1}{N} \sum_{n=1}^{N} \frac{Z_1^n(\beta)}{Z_N(\beta)} \int_{-\infty}^{\infty} d^{N-1}x_b \; \underbrace{\varrho_1(x_{1b}, x_{2b}; t) \cdots \varrho_1(x_{Nb}, x_{1b'}; t)}_{\text{broken } n\text{-cycle}} \; \underbrace{Z_{N-n}(\beta)}_{(N-n)\text{-cycle}}$$

- Integrating broken cycles: Master integral

$$Z_1(\beta) Z_1(\beta') \int_{-\infty}^{\infty} dx_{2b} \; \varrho_1(x_{1b}, x_{2b}; t, \beta) \varrho_1(x_{2b}, x_{1b'}; t, \beta') = Z_1(\beta + \beta') \; \varrho_1\left(x_{1b}, x_{1b'}; t, (\beta + \beta')\right)$$

- One-particle reduced density matrix

$$\varrho_1^{(\mathrm{r})}(x_{1b}, x_{1b'}; t) = \frac{1}{N} \frac{1}{Z_N(\beta)} \sum_{n=1}^{N} \varrho_1(x_{1b}, x_{1b'}; t, n\beta) \; Z_1(n\beta) \; Z_{N-n}(\beta)$$

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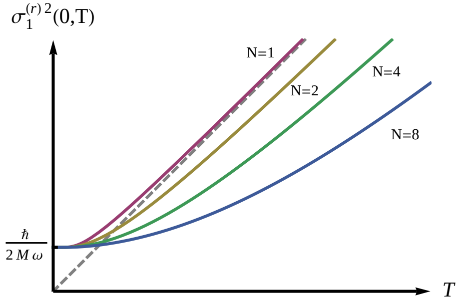
Reduced One-Particle Width

- Equilibrium reduced one-particle width

$$\sigma_1^{(\mathrm{r})2}(0,\beta) = \frac{1}{N}\frac{1}{Z_N(\beta)}\sum_{n=1}^{N}\sigma_1^2(0,n\beta)\, Z_1(n\beta)\, Z_{N-n}(\beta)$$

- Time dependence factorizes $\sigma_1^{(\mathrm{r})2}(t,T) = \sigma_1^{(\mathrm{r})2}(0,T)\, f(t)$
- Oscillating time dependence

$$f(t) = \frac{1}{2}\left(1+\frac{\omega^2}{\Omega^2}\right) + \frac{1}{2}\left(1-\frac{\omega^2}{\Omega^2}\right)\cos(2\Omega t)$$



Outlook

- Comparison with Grand-Canonical Ensemble
- Quench with interactions:
  Oscillation? Damping? Emergence of collective motion
- Quench: Interaction between two BECs