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Pismeni deo ispita iz KVANTNE STATISTIČKE FIZIKE

1. Posmatrajmo jednodimenzionalni harmonijski oscilator mase m i frekvencije ω koji je u ravnoteži sa termostatom temperature T.

   (a) Koristeći klasičnu Maxwell–Boltzmannovu statistiku, izračunati neoređenost $(\Delta x)_{\mathrm{cl}}$ koordinate oscilatora. Neodređenost koordinate je definisana kao

   $$(\Delta x)_{\mathrm{cl}} = \sqrt{\langle x^2\rangle_{\mathrm{cl}} - \langle x\rangle_{\mathrm{cl}}^2},$$

   gde je x (klasična) koordinata, dok $\langle \ldots \rangle_{\mathrm{cl}}$ označava usrednjavanje po Maxwell–Boltzmannovoj statistici.

   (b) Koristeći kvantnu statistiku, izračunati neodređenost $(\Delta x)_{\mathrm{qu}}$ koordinate oscilatora.

   (c) Izračunati veličinu

   $$\delta = \frac{(\Delta x)_{\mathrm{cl}} - (\Delta x)_{\mathrm{qu}}}{(\Delta x)_{\mathrm{qu}}}$$

   i ispitati njeno ponašanje u graničnim slučajevima visokih ($k_B T \gg \hbar\omega$) i niskih ($k_B T \ll \hbar\omega$) temperatura. Rezultat u graničnim slučajevima treba da sadrži ne samo odgovarajuću graničnu vrednost, već i prvu temperatursku korekciju u relevantnoj oblasti temperatura.

2. Posmatrajmo elektronski gas koji se sastoji od N elektrona koji se nalaze u trodimenzionalnom sudu zapremine V. Zavisnost jednočestične energije ε od intenziteta impulsa p je ultrarelativističkog tipa, $\varepsilon_p = cp$, gde je c brzina svetlosti.

   (a) Izračunati jednočestičnu gustinu stanja $g(\varepsilon)$.

   (b) Izračunati Fermijevu energiju ε_F.

   (c) Odrediti temperatursku zavisnost hemijskog potencijala, $\mu(T)$, i unutrašnje energije, $U(T)$, gasa u oblasti niskih temperatura T ($k_B T \ll \varepsilon_F$). Zadržati se na članovima najnižeg (nenultog) reda po maloj veličini $k_B T/\varepsilon_F$.

   (d) Odrediti toplotni kapacitet C_V na niskim temperaturama T.

3. Hamiltonijan slabo neidealnog Bose gasa koji se sastoji od N bozona spina 0 u zapremini V je oblika

   $$\hat{H} = \frac{2\pi\hbar^2 a}{m}\frac{N^2}{V}\left(1 + \frac{4\pi\hbar^2 a}{V}\sum_{\mathbf{p}\neq 0}\frac{1}{\mathbf{p}^2}\right) + \sum_{\mathbf{p}\neq 0}\frac{\mathbf{p}^2}{2m}\,\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} + \frac{2\pi\hbar^2 a}{m}\frac{N}{V}\sum_{\mathbf{p}\neq 0}\left(\hat{a}_{\mathbf{p}}\hat{a}_{-\mathbf{p}} + \hat{a}^\dagger_{-\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}} + 2\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}}\right),$$

   gde je m masa pojedinačne čestice gasa, a je dužina rasejanja (koja zavisi od potencijala parne interakcije među česticama), dok operatori $\hat{a}^\dagger_{\mathbf{p}}$ ($\hat{a}_{\mathbf{p}}$) kreiraju (anihiliraju) jednu česticu impulsa $\mathbf{p}$. Prelazeći na operatore $\hat{b}_{\mathbf{p}}, \hat{b}^\dagger_{\mathbf{p}}$ ($\mathbf{p} \neq 0$) relacijama

   $$\hat{a}_{\mathbf{p}} = \mathrm{ch}(\theta_p)\,\hat{b}_{\mathbf{p}} + \mathrm{sh}(\theta_p)\,\hat{b}^\dagger_{-\mathbf{p}}, \qquad \hat{a}^\dagger_{-\mathbf{p}} = \mathrm{sh}(\theta_p)\,\hat{b}_{\mathbf{p}} + \mathrm{ch}(\theta_p)\,\hat{b}^\dagger_{-\mathbf{p}},$$

   (a) naći uslov koji treba da zadovoljavaju parametri θ_p da bi se Hamiltonijan $\hat{H}$ dijagonalizovao,

   (b) koristeći dobijeni rezultat, izračunati relativni broj čestica van kondenzata na nuli temperature i izraziti ga u funkciji tzv. gasnog parametra $\frac{Na^3}{V} \ll 1$.

   Da biste uprostili izraze, možete koristiti da je kvadrat brzine zvuka na $T = 0$ $u^2 = \frac{4\pi\hbar^2 aN}{m^2 V}$.

Zadatke pripremio
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1. (a) $$\langle x\rangle_{cl} = \frac{\int \frac{dx\,dp}{2\pi\hbar}\, x\, e^{-\beta\mathcal{H}}}{\int \frac{dx\,dp}{2\pi\hbar}\, e^{-\beta\mathcal{H}}} \quad , \quad \mathcal{H} = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2$$

$$\boxed{\int_{-\infty}^{+\infty} dx\; e^{-ax^2} = \sqrt{\frac{\pi}{a}} \quad (a>0)}$$

$$\frac{d}{da}\int_{-\infty}^{+\infty} dx\; e^{-ax^2} = \int_{-\infty}^{+\infty} dx\,(-x^2)\,e^{-ax^2} = \sqrt{\pi}\left(-\frac{1}{2}\right)a^{-3/2}$$

$$\boxed{\int_{-\infty}^{+\infty} dx\; x^2\, e^{-ax^2} = \frac{\sqrt{\pi}}{2}\,a^{-3/2}}$$

$$\langle x\rangle_{cl} = \frac{\int_{-\infty}^{+\infty} dx\; x\,\exp\left(-\frac{\beta m\omega^2}{2}x^2\right)}{\int_{-\infty}^{+\infty} dx\;\exp\left(-\frac{\beta m\omega^2}{2}x^2\right)} = \boxed{0 = \langle x\rangle_{cl}}$$

$$\langle x^2\rangle_{cl} = \frac{\int_{-\infty}^{+\infty} dx\; x^2\,\exp\left(-\frac{\beta m\omega^2}{2}x^2\right)}{\int_{-\infty}^{+\infty} dx\;\exp\left(-\frac{\beta m\omega^2}{2}x^2\right)} = \frac{\frac{\sqrt{\pi}}{2}\left(\frac{\beta m\omega^2}{2}\right)^{-3/2}}{\sqrt{\pi}\left(\frac{\beta m\omega^2}{2}\right)^{-1/2}} = \frac{1}{2}\cdot\frac{2}{\beta m\omega^2} = \boxed{\frac{k_BT}{m\omega^2} = \langle x^2\rangle_{cl}}$$

$$(\Delta x)_{cl} = \sqrt{\langle x^2\rangle_{cl} - \langle x\rangle_{cl}^2} \quad , \quad \boxed{(\Delta x)_{cl} = \left(\frac{k_BT}{m\omega^2}\right)^{1/2}}$$

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(b) $$\boxed{\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{b}^\dagger + \hat{b}\right)} \quad , \quad \hat{\rho} = \frac{1}{Z}\,e^{-\beta\hbar\omega\left(\hat{b}^\dagger\hat{b} + \frac{1}{2}\right)}$$

$$\langle\hat{x}\rangle_{qu} = 0, \quad \langle\hat{x}^2\rangle_{qu} = \frac{\hbar}{2m\omega}\left\langle(\hat{b}^\dagger+\hat{b})(\hat{b}^\dagger+\hat{b})\right\rangle_{qu} = \frac{\hbar}{2m\omega}\left(\langle\hat{b}^\dagger\hat{b}\rangle_{qu} + \langle\hat{b}\hat{b}^\dagger\rangle_{qu}\right)$$

$$= \frac{\hbar}{2m\omega}\left(\frac{2}{e^{\beta\hbar\omega}-1} + 1\right) = \frac{\hbar}{2m\omega}\,\frac{2+e^{\beta\hbar\omega}-1}{e^{\beta\hbar\omega}-1} = \frac{\hbar}{2m\omega}\,\frac{e^{\beta\hbar\omega}+1}{e^{\beta\hbar\omega}-1} = \boxed{\frac{\hbar}{2m\omega}\,\mathrm{cth}\left(\frac{\beta\hbar\omega}{2}\right) = \langle\hat{x}^2\rangle_{qu}}$$

$$\boxed{(\Delta x)_{qu} = \left(\frac{\hbar}{2m\omega}\,\mathrm{cth}\left(\frac{\hbar\omega}{2k_BT}\right)\right)^{1/2}}$$

$$\mathrm{cth}\left(\frac{x}{2}\right) = \frac{e^x+1}{e^x-1} = \frac{1+e^{-x}}{1-e^{-x}}$$

(c) $$\delta = \frac{(\Delta x)_{cl} - (\Delta x)_{qu}}{(\Delta x)_{qu}} = \frac{\left(\frac{k_BT}{m\omega^2}\right)^{1/2} - \left(\frac{\hbar}{2m\omega}\,\mathrm{cth}\left(\frac{\hbar\omega}{2k_BT}\right)\right)^{1/2}}{\left(\frac{\hbar}{2m\omega}\,\mathrm{cth}\left(\frac{\hbar\omega}{2k_BT}\right)\right)^{1/2}}$$

$\underline{x \ll 1}$ $$\mathrm{cth}\left(\frac{x}{2}\right) = \frac{1+1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\dots}{1-1+x-\frac{x^2}{2!}+\frac{x^3}{3!}+\dots}$$

$$\mathrm{cth}\left(\frac{x}{2}\right) = \frac{2-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\dots}{x\left(1-\frac{x}{2!}+\frac{x^2}{3!}-\dots\right)}$$

(c1) $k_BT \gg \hbar\omega$

$$\mathrm{cth}\left(\frac{x}{2}\right) = \frac{2}{x}\,\frac{1-\frac{x}{2}+\frac{x^2}{4}-\frac{x^3}{12}+\dots}{1-\frac{x}{2}+\frac{x^2}{6}-\dots} = \frac{2}{x}\left(1-\frac{x}{2}+\frac{x^2}{4}-\dots\right)\left(1+\frac{x}{2}\underbrace{-\frac{x^2}{6}+\frac{x^2}{4}}_{x^2\left(\frac{1}{4}-\frac{1}{6}\right) = \frac{x^2}{12}}+\dots\right)$$

$$= \frac{2}{x}\left(1+\cancel{\frac{x}{2}}+\frac{x^2}{12}-\cancel{\frac{x}{2}}-\cancel{\frac{x^2}{4}}+\cancel{\frac{x^2}{4}}+\dots\right)$$

$$\boxed{\mathrm{cth}\left(\frac{x}{2}\right) \simeq \frac{2}{x} + \frac{x}{6} \quad , \quad x \ll 1}$$

$$\delta = \frac{\left(\frac{k_BT}{m\omega^2}\right)^{1/2} - \left(\frac{\hbar}{2m\omega}\left(\frac{2k_BT}{\hbar\omega}+\frac{\hbar\omega}{6k_BT}\right)\right)^{1/2}}{\left(\frac{\hbar}{2m\omega}\left(\frac{2k_BT}{\hbar\omega}+\frac{\hbar\omega}{6k_BT}\right)\right)^{1/2}} = \frac{\left(\frac{k_BT}{m\omega^2}\right)^{1/2} - \left(\frac{\hbar}{2m\omega}\,\frac{2k_BT}{\hbar\omega}\right)^{1/2}\left(1+\left(\frac{\hbar\omega}{k_BT}\right)^2\frac{1}{12}\right)^{1/2}}{\left(\frac{\hbar}{2m\omega}\,\frac{2k_BT}{\hbar\omega}\right)^{1/2}\left(1+\frac{1}{12}\left(\frac{\hbar\omega}{k_BT}\right)^2\right)^{1/2}}$$

$$\delta = \frac{1-\left(1+\frac{1}{12}\left(\frac{\hbar\omega}{k_BT}\right)^2\right)^{1/2}}{\left(1+\frac{1}{12}\left(\frac{\hbar\omega}{k_BT}\right)^2\right)^{1/2}} \quad , \quad \boxed{\delta \approx -\frac{1}{24}\left(\frac{\hbar\omega}{k_BT}\right)^2}$$

(c2) $k_BT \ll \hbar\omega$, $$\mathrm{cth}\left(\frac{x}{2}\right) = \frac{1+e^{-x}}{1-e^{-x}} \cong 1+2e^{-x}, \quad x \to +\infty$$

$$\delta = \frac{\left(\frac{k_BT}{m\omega^2}\right)^{1/2} - \left(\frac{\hbar}{2m\omega}\left(1+2e^{-\beta\hbar\omega}\right)\right)^{1/2}}{\left(\frac{\hbar}{2m\omega}\left(1+2e^{-\beta\hbar\omega}\right)\right)^{1/2}} = \frac{\cancel{\left(\frac{\hbar}{2m\omega}\right)^{1/2}}\left(\left(\frac{2m\omega}{\hbar}\,\frac{k_BT}{m\omega^2}\right)^{1/2} - \left(1+2e^{-\beta\hbar\omega}\right)^{1/2}\right)}{\cancel{\left(\frac{\hbar}{2m\omega}\right)^{1/2}}\left(1+2e^{-\beta\hbar\omega}\right)^{1/2}}$$

$$\approx \frac{\sqrt{\frac{2}{\beta\hbar\omega}} - \left(1+e^{-\beta\hbar\omega}\right)}{1+e^{-\beta\hbar\omega}} \quad , \quad \boxed{\delta \approx \sqrt{\frac{2k_BT}{\hbar\omega}} - 1}$$

2. (a) $g(\varepsilon) = \sum_{\vec{p}\sigma} \delta(\varepsilon - \varepsilon_p) = 2\frac{V}{(2\pi\hbar)^3} \times 4\pi \int_0^{+\infty} dp\, p^2 \frac{1}{c}\,\delta\!\left(p - \frac{\varepsilon}{c}\right) = \frac{V}{\pi^2\hbar^3}\frac{1}{c}\left(\frac{\varepsilon}{c}\right)^2\theta(\varepsilon)$

$$\boxed{g(\varepsilon) = \frac{1}{\pi^2}\frac{V}{(\hbar c)^3}\varepsilon^2\theta(\varepsilon)}$$

(b) $N = \int_{-\infty}^{+\infty} d\varepsilon\, \bar{n}(\varepsilon)\, g(\varepsilon)\,, \quad \bar{n}(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)}+1}$

$\underline{T=0}$ $\quad N = \int_{-\infty}^{+\infty} d\varepsilon\, \theta(\varepsilon_F - \varepsilon)\frac{1}{\pi^2}\frac{V}{(\hbar c)^3}\varepsilon^2\theta(\varepsilon) = \frac{1}{\pi^2}\frac{V}{(\hbar c)^3}\int_0^{\varepsilon_F} d\varepsilon\,\varepsilon^2 = \frac{1}{\pi^2}\frac{V}{(\hbar c)^3}\frac{\varepsilon_F^3}{3} = \frac{V}{3\pi^2}\left(\frac{\varepsilon_F}{\hbar c}\right)^3$

$$\left(\frac{\varepsilon_F}{\hbar c}\right)^3 = 3\pi^2\frac{N}{V}\,, \quad \boxed{\varepsilon_F = \hbar c\left(3\pi^2\frac{N}{V}\right)^{1/3}}$$

(c) $\underline{k_B T \ll \varepsilon_F}$ $\quad N = \int_{-\infty}^{+\infty} d\varepsilon\,\frac{1}{e^{\beta(\varepsilon-\mu)}+1}\frac{1}{\pi^2}\frac{V}{(\hbar c)^3}\varepsilon^2\theta(\varepsilon) = \frac{1}{\pi^2}\frac{V}{(\hbar c)^3}\int_0^{+\infty} d\varepsilon\,\frac{\varepsilon^2}{e^{\beta(\varepsilon-\mu)}+1} = \left\{\begin{matrix}\beta\varepsilon = x & d\varepsilon = \frac{dx}{\beta}\\ \beta\mu = \xi & \end{matrix}\right\}$

$$= \frac{1}{\pi^2}\frac{V}{(\hbar c)^3}\frac{1}{\beta^3}\int_0^{+\infty} dx\,\frac{x^2}{e^{x-\xi}+1}$$

$\underline{\text{za } \xi \gg 1}$: $\int_0^{+\infty} dx\,\frac{x^2}{e^{x-\xi}+1} \simeq \int_0^{\xi} dx\, x^2 + \frac{\pi^2}{6}\frac{d}{dx}(x^2)\Big|_{x=\xi} + \ldots = \frac{\xi^3}{3} + \frac{\pi^2}{6}\cdot 2\xi + \ldots$

$$= \frac{\xi^3}{3}\left(1 + \frac{3\cdot 2\pi^2}{6}\xi^{-2} + \ldots\right) = \frac{\xi^3}{3}\left(1 + \pi^2\xi^{-2} + \ldots\right)$$

$$N = \frac{1}{\pi^2}\frac{V}{(\hbar c)^3}(k_B T)^3 \cdot \frac{1}{3}\left(\frac{\mu}{k_B T}\right)^3\left[1 + \pi^2\left(\frac{k_B T}{\mu}\right)^2 + \ldots\right]$$

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$$\frac{V}{3\pi^2}\left(\frac{\varepsilon_F}{\hbar c}\right)^3 = \frac{V}{3\pi^2}\frac{1}{(\hbar c)^3}(k_B T)^3\left(\frac{\mu}{k_B T}\right)^3\left[1 + \pi^2\left(\frac{k_B T}{\mu}\right)^2 + \ldots\right]$$

$$\varepsilon_F^3 = \mu^3\left[1 + \pi^2\left(\frac{k_B T}{\mu}\right)^2 + \ldots\right]$$

$$\varepsilon_F = \mu\left[1 + \pi^2\left(\frac{k_B T}{\mu}\right)^2 + \ldots\right]^{1/3} \simeq \mu\left[1 + \frac{\pi^2}{3}\left(\frac{k_B T}{\mu}\right)^2 + \ldots\right]$$

$$\mu = \varepsilon_F\left[A_1 + A_2\left(\frac{k_B T}{\varepsilon_F}\right)^2 + \ldots\right] \longrightarrow \varepsilon_F = \varepsilon_F\left[A_1 + A_2\left(\frac{k_B T}{\varepsilon_F}\right)^2 + \ldots\right]\left[1 + \frac{\pi^2}{3}\overbrace{\left(\frac{k_B T}{\varepsilon_F}\right)^2\left(A_1 + A_2\left(\frac{k_B T}{\varepsilon_F}\right)^2 + \ldots\right)^{-2}}^{\approx A_1} + \ldots\right]$$

$$1 = A_1 + \frac{\pi^2}{3}\left(\frac{k_B T}{\varepsilon_F}\right)^2 A_1^2 + A_2\left(\frac{k_B T}{\varepsilon_F}\right)^2 + \ldots \quad \Rightarrow \quad \boxed{A_1 = 1,\ A_2 = -\frac{\pi^2}{3}}$$

$$\boxed{\mu(T) = \varepsilon_F\left[1 - \frac{\pi^2}{3}\left(\frac{k_B T}{\varepsilon_F}\right)^2 + \ldots\right],\ k_B T \ll \varepsilon_F}$$

$$U = \int_{-\infty}^{+\infty} d\varepsilon\,\varepsilon\,\bar{n}(\varepsilon)\,g(\varepsilon) = \int_0^{+\infty} d\varepsilon\,\varepsilon\,\frac{1}{e^{\beta(\varepsilon-\mu)}+1}\frac{1}{\pi^2}\frac{V}{(\hbar c)^3}\varepsilon^2 = \frac{1}{\pi^2}\frac{V}{(\hbar c)^3}\int_0^{+\infty} d\varepsilon\,\frac{\varepsilon^3}{e^{\beta(\varepsilon-\mu)}+1}$$

$$= \frac{1}{\pi^2}\frac{V}{(\hbar c)^3}(k_B T)^4\int_0^{+\infty} dx\,\frac{x^3}{e^{x-\xi}+1} = \frac{1}{\pi^2}k_B T\, V\left(\frac{k_B T}{\hbar c}\right)^3\left[\int_0^{\xi} dx\, x^3 + \frac{\pi^2}{6}\frac{d}{dx}(x^3)\Big|_{x=\xi} + \ldots\right]$$

$$= \frac{1}{\pi^2}k_B T\, V\left(\frac{k_B T}{\hbar c}\right)^3\left[\frac{1}{4}\left(\frac{\mu}{k_B T}\right)^4 + \frac{\pi^2}{6}\cdot 3\left(\frac{\mu}{k_B T}\right)^2 + \ldots\right]$$

$$= \frac{1}{\pi^2}k_B T\, V\left(\frac{k_B T}{\hbar c}\right)^3\frac{1}{4}\left(\frac{\mu}{k_B T}\right)^4\left[1 + 4\frac{\pi^2}{6}\cdot 3\left(\frac{k_B T}{\mu}\right)^2 + \ldots\right]$$

$$= \frac{1}{\pi^2}\cancel{k_B T}\, V\left(\frac{\cancel{k_B T}}{\hbar c}\right)^3\frac{1}{4}\left(\frac{\varepsilon_F}{\cancel{k_B T}}\right)^4\left[1 - \frac{\pi^2}{3}\left(\frac{k_B T}{\varepsilon_F}\right)^2 + \ldots\right]^4\left[1 + 2\pi^2\left(\frac{k_B T}{\varepsilon_F}\right)^2\left(1 - \frac{\pi^2}{3}\left(\frac{k_B T}{\varepsilon_F}\right)^2 + \ldots\right)^{-2}\right]$$

$$= \frac{1}{\cancel{\pi^2}}\frac{V}{\cancel{(\hbar c)^3}}\cdot\frac{1}{4}\cdot\varepsilon_F\cdot\cancel{(\hbar c)^3}\,3\cancel{\pi^2}\frac{N}{V}\left[1 - \frac{4\pi^2}{3}\left(\frac{k_B T}{\varepsilon_F}\right)^2 + \ldots\right]\left[1 + 2\pi^2\left(\frac{k_B T}{\varepsilon_F}\right)^2 + \ldots\right]$$

$$= \frac{3}{4}N\varepsilon_F\left[1 + \left(2 - \frac{4}{3}\right)\pi^2\left(\frac{k_B T}{\varepsilon_F}\right)^2 + \ldots\right] = \underbrace{\frac{3}{4}N\varepsilon_F}_{\parallel}\left[1 + \frac{2}{3}\pi^2\left(\frac{k_B T}{\varepsilon_F}\right)^2 + \ldots\right]$$

$$U(T{=}0) = \int_0^{\varepsilon_F} d\varepsilon\,\frac{1}{\pi^2}\frac{V}{(\hbar c)^3}\varepsilon^3 = \frac{1}{\cancel{\pi^2}}\frac{V}{\cancel{(\hbar c)^3}}\frac{1}{4}\varepsilon_F\cdot\cancel{(\hbar c)^3}\,3\cancel{\pi^2}\frac{N}{V} = \boxed{\frac{3}{4}N\varepsilon_F = U(T{=}0)}$$

(d) $C_V = \left(\frac{dU}{dT}\right)_V$

$k_BT \ll \varepsilon_F$ $\quad C_V = \frac{3}{\cancel{4}} N \varepsilon_F \frac{\cancel{2}}{3}\pi^2 \left(\frac{k_B}{\varepsilon_F}\right)^2 \cdot \cancel{2}T$

$$\boxed{C_V = Nk_B \cdot \pi^2 \frac{k_BT}{\varepsilon_F}}$$

$$C_V = Nk_B \cdot \pi^2 \frac{k_BT}{\hbar c\,(3\pi^2)^{1/3}}\left(\frac{V}{N}\right)^{1/3}$$

$$= Nk_B \frac{\pi^2}{(3\pi^2)^{1/3}} \frac{k_BT}{\hbar c}\left(\frac{V}{N}\right)^{1/3}$$

$$= Nk_B \frac{3^{2/3}\pi^2}{3\cdot \pi^{2/3}} \frac{k_BT}{\hbar c}\left(\frac{V}{N}\right)^{1/3}$$

$$= \boxed{Nk_B \frac{(3\pi^2)^{2/3}}{3} \frac{k_BT}{\hbar c}\left(\frac{V}{N}\right)^{1/3} = C_V}$$

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3. $\hat{\mathcal{H}} = const + \sum_{\vec p\neq 0} \xi_p \hat a^\dagger_{\vec p}\hat a_{\vec p} + \frac{mu^2}{2}\sum_{\vec p\neq 0}\left(\hat a_{\vec p}\hat a_{-\vec p} + \hat a^\dagger_{\vec p}\hat a^\dagger_{-\vec p} + 2\hat a^\dagger_{\vec p}\hat a_{\vec p}\right)$

(a)

$$\hat a^\dagger_{\vec p}\hat a_{\vec p} = \left(\mathrm{sh}\,\theta_p \hat b_{-\vec p} + \mathrm{ch}\,\theta_p \hat b^\dagger_{\vec p}\right)\left(\mathrm{ch}\,\theta_p \hat b_{\vec p} + \mathrm{sh}\,\theta_p \hat b^\dagger_{-\vec p}\right) = \frac12 \mathrm{sh}(2\theta_p)\left(\hat b_{-\vec p}\hat b_{\vec p} + \hat b^\dagger_{\vec p}\hat b^\dagger_{-\vec p}\right) + \mathrm{sh}^2\theta_p\, \hat b_{-\vec p}\hat b^\dagger_{-\vec p} + \mathrm{ch}^2\theta_p\, \hat b^\dagger_{\vec p}\hat b_{\vec p}$$

$$\boxed{\hat a^\dagger_{\vec p}\hat a_{\vec p} = \frac12 \mathrm{sh}(2\theta_p)\left(\hat b_{-\vec p}\hat b_{\vec p} + \hat b^\dagger_{\vec p}\hat b^\dagger_{-\vec p}\right) + \mathrm{sh}^2\theta_p + \mathrm{sh}^2\theta_p\, \hat b^\dagger_{-\vec p}\hat b_{-\vec p} + \mathrm{ch}^2\theta_p\, \hat b^\dagger_{\vec p}\hat b_{\vec p}}$$

$$\hat a_{\vec p}\hat a_{-\vec p} = \left(\mathrm{ch}\,\theta_p \hat b_{\vec p} + \mathrm{sh}\,\theta_p \hat b^\dagger_{-\vec p}\right)\left(\mathrm{ch}\,\theta_p \hat b_{-\vec p} + \mathrm{sh}\,\theta_p \hat b^\dagger_{\vec p}\right)$$

$$= \mathrm{ch}^2\theta_p\, \hat b_{\vec p}\hat b_{-\vec p} + \frac12 \mathrm{sh}(2\theta_p)\left(1 + \hat b^\dagger_{\vec p}\hat b_{\vec p}\right) + \frac12\mathrm{sh}(2\theta_p)\,\hat b^\dagger_{-\vec p}\hat b_{-\vec p} + \mathrm{sh}^2\theta_p\, \hat b^\dagger_{-\vec p}\hat b^\dagger_{\vec p}$$

$$\boxed{\hat a_{\vec p}\hat a_{-\vec p} = \mathrm{ch}^2\theta_p\, \hat b_{\vec p}\hat b_{-\vec p} + \mathrm{sh}^2\theta_p\, \hat b^\dagger_{-\vec p}\hat b^\dagger_{\vec p} + \frac12\mathrm{sh}(2\theta_p)\left[1 + \hat b^\dagger_{\vec p}\hat b_{\vec p} + \hat b^\dagger_{-\vec p}\hat b_{-\vec p}\right]}$$

$$\boxed{\hat a^\dagger_{-\vec p}\hat a^\dagger_{\vec p} = \mathrm{ch}^2\theta_p\, \hat b^\dagger_{-\vec p}\hat b^\dagger_{\vec p} + \mathrm{sh}^2\theta_p\, \hat b_{\vec p}\hat b_{-\vec p} + \frac12\mathrm{sh}(2\theta_p)\left[1 + \hat b^\dagger_{\vec p}\hat b_{\vec p} + \hat b^\dagger_{-\vec p}\hat b_{-\vec p}\right]}$$

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vandijagonalni deo

$$\sum_{\vec p\neq\vec 0}\Big[\xi_p \frac12 \mathrm{sh}(2\theta_p)\left(\hat b_{-\vec p}\hat b_{\vec p} + \hat b^\dagger_{\vec p}\hat b^\dagger_{-\vec p}\right) + \frac{mu^2}{2}\underbrace{\left(\mathrm{ch}^2\theta_p + \mathrm{sh}^2\theta_p\right)}_{\mathrm{ch}(2\theta_p)}\left(\hat b_{\vec p}\hat b_{-\vec p} + \hat b^\dagger_{-\vec p}\hat b^\dagger_{\vec p}\right) + \frac{mu^2}{2}\cdot 2\cdot\frac12\mathrm{sh}(2\theta_p)\left(\hat b_{-\vec p}\hat b_{\vec p} + \hat b^\dagger_{\vec p}\hat b^\dagger_{-\vec p}\right)\Big]$$

$\Rightarrow$ uslov dijagonalizacije $\quad \xi_p\cdot\frac12\mathrm{sh}(2\theta_p) + \frac{mu^2}{2}\mathrm{ch}(2\theta_p) + \frac{mu^2}{2}\mathrm{sh}(2\theta_p) = 0$

$$(\xi_p + mu^2)\,\mathrm{sh}(2\theta_p) = -mu^2\,\mathrm{ch}(2\theta_p)$$

$$\boxed{\mathrm{th}(2\theta_p) = -\frac{mu^2}{\xi_p + mu^2}}$$

(b) $\hat N = \sum_{\vec p}\hat a^\dagger_{\vec p}\hat a_{\vec p} = \hat a^\dagger_0\hat a_0 + \sum_{\vec p\neq\vec 0}\hat a^\dagger_{\vec p}\hat a_{\vec p}$

$$\langle \hat N_{\vec p\neq\vec 0}\rangle = \sum_{\vec p\neq\vec 0}\langle \hat a^\dagger_{\vec p}\hat a_{\vec p}\rangle = \sum_{\vec p\neq\vec 0}\frac12\mathrm{sh}(2\theta_p)\left(\langle \hat b_{-\vec p}\hat b_{\vec p}\rangle + \langle \hat b^\dagger_{\vec p}\hat b^\dagger_{-\vec p}\rangle\right) + \sum_{\vec p\neq\vec 0}\mathrm{sh}^2\theta_p + \sum_{\vec p\neq\vec 0}\left(\mathrm{sh}^2\theta_p\langle \hat b^\dagger_{-\vec p}\hat b_{-\vec p}\rangle + \mathrm{ch}^2\theta_p\langle \hat b^\dagger_{\vec p}\hat b_{\vec p}\rangle\right)$$

na $T=0$ $\quad \langle \hat b_{-\vec p}\hat b_{\vec p}\rangle = \langle \hat b^\dagger_{\vec p}\hat b^\dagger_{-\vec p}\rangle = \langle \hat b^\dagger_{\vec p}\hat b_{\vec p}\rangle = 0$

$$\langle \hat N_{\vec p\neq\vec 0}\rangle = \sum_{\vec p\neq 0}\mathrm{sh}^2\theta_p = \sum_{\vec p\neq\vec 0}\frac12\left(\mathrm{ch}(2\theta_p) - 1\right) = \sum_{\vec p\neq\vec 0}\frac12\left(\frac{1}{\sqrt{1-\mathrm{th}^2(2\theta_p)}} - 1\right)$$

$$\langle \hat N_{\vec p\neq\vec 0}\rangle = \frac{V}{(2\pi\hbar)^3}\times 4\pi\int_0^{+\infty}dp\,p^2\,\frac12\left(\frac{\xi_p + mu^2}{\sqrt{(\xi_p+mu^2)^2 - (mu^2)^2}} - 1\right) = \left\{\begin{array}{ll}\xi = \frac{p^2}{2m} & dp = \frac{\sqrt{2m}\,d\xi}{2\sqrt{\xi}}\\ p = \sqrt{2m}\sqrt{\xi} & \end{array}\right\}$$

$$= \frac{V}{4\pi^2\hbar^3}\int_0^{+\infty}\frac{\sqrt{2m}}{2}\frac{d\xi}{\sqrt\xi}\,2m\xi\left(\frac{\xi + mu^2}{\sqrt{\xi(\xi + 2mu^2)}} - 1\right)$$

$$= \frac{V(2m)^{3/2}}{2^3\pi^2\hbar^3}\int_0^{+\infty}d\xi\left(\frac{\xi + 2mu^2 - mu^2}{\sqrt{\xi + 2mu^2}} - \sqrt{\xi}\right) = \frac{Vm^{3/2}}{2^{3/2}\pi^2\hbar^3}\underbrace{\int_0^{+\infty}d\xi\left(\sqrt{\xi+2mu^2} - \frac{mu^2}{\sqrt{\xi+2mu^2}} - \sqrt{\xi}\right)}_{I}$$

$$\int_0^{+\infty} d\xi \, \sqrt{\xi + 2mu^2} = \frac{2}{3}\left(\xi + 2mu^2\right)^{3/2}$$

$$\int_0^{+\infty} \frac{d\xi}{\sqrt{\xi + 2mu^2}} = 2\left(\xi + 2mu^2\right)^{1/2}$$

$$I = \left(\frac{2}{3}\left(\xi + 2mu^2\right)^{3/2} - 2mu^2\left(\xi + 2mu^2\right)^{1/2} - \frac{2}{3}\xi^{3/2}\right)_{\xi=0}^{+\infty}$$

$\underline{\xi \to +\infty}$

$$\frac{2}{3}\xi^{3/2}\left(1 + \frac{2mu^2}{\xi}\right)^{3/2} - 2mu^2\,\xi^{1/2}\left(1 + \frac{2mu^2}{\xi}\right)^{1/2} - \frac{2}{3}\xi^{3/2}$$

$$= \frac{2}{3}\xi^{3/2}\left[1 + \frac{3}{2}\frac{2mu^2}{\xi} + O\left(\frac{1}{\xi^2}\right)\right] - 2mu^2\,\xi^{1/2}\left(1 + O\left(\frac{1}{\xi}\right)\right) - \frac{2}{3}\xi^{3/2}$$

$$= \frac{2}{3}\xi^{3/2} + 2mu^2\,\xi^{1/2} + O\left(\frac{1}{\sqrt{\xi}}\right) - 2mu^2\,\xi^{1/2} + O\left(\frac{1}{\sqrt{\xi}}\right) - \frac{2}{3}\xi^{3/2} = O\left(\frac{1}{\sqrt{\xi}}\right), \; \xi \to +\infty$$

$$\Rightarrow \boxed{I = -\left(\frac{2}{3}\left(2mu^2\right)^{3/2} - 2mu^2\left(2mu^2\right)^{1/2}\right) = \frac{1}{3}\left(2mu^2\right)^{3/2}}$$

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$$\langle \hat{N}_{\vec{p}\neq\vec{0}} \rangle = \frac{V m^{3/2}}{\cancel{2^{3/2}}\,\pi^2 \hbar^3} \cdot \frac{1}{3} \cdot \cancel{2^{3/2}}\, m^{3/2}\, u^3$$

$$= \frac{\cancel{V} m^3}{3\pi^2 \hbar^3}\,\frac{4\pi\hbar^2 a}{m^2}\,\frac{N}{\cancel{V}}\sqrt{\frac{4\pi\hbar^2 a}{m^2}\,\frac{N}{V}}$$

$$\frac{\langle \hat{N}_{\vec{p}\neq\vec{0}} \rangle}{N} = \frac{\cancel{m^3}}{3\pi^2 \cancel{\hbar^3}}\,\frac{4\pi \cancel{\hbar^2} a}{\cancel{m^2}}\,\frac{2\sqrt{\pi}\,\cancel{\hbar}}{\cancel{m}}\sqrt{a\,\frac{N}{V}}\,, \qquad \boxed{\frac{\langle \hat{N}_{\vec{p}\neq\vec{0}} \rangle}{N} = \frac{8}{3\sqrt{\pi}}\sqrt{\frac{N}{V}a^3}}$$